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On the Whirling and Vibration of Shafts

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VIII. *On the Whirling and Vibration of Shafts.*

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CHAPTER I.—INTRODUCTION AND DESCRIPTION OF EXPERIMENTAL APPARATUS.

INTRODUCTION.

1. It is well known that every shaft, *however nearly balanced*, when driven at a particular speed, bends, and, unless the amount of deflection be limited, might even break, although at higher speeds the shaft again runs true. The particular or "critical" speed depends on the manner in which the shaft is supported, its size and modulus of elasticity, and the size, weight, and position of any pulleys it carries.

The theory for the case of an unloaded shaft first received attention at the hands of Professor RANKINE,* who obtained numerical formulæ for the cases of an unloaded shaft resting freely on a bearing at each end, and for an overhanging shaft working in a shoulder at one end.

Professor GREENHILL has also obtained formulæ for the cases of an unloaded shaft resting on bearings at each end, and fixed in direction at each end.†

The theory has been further extended to the case of a shaft loaded with pulleys, by Professor REYNOLDS; and the object of this investigation is to apply that theory and so obtain formulæ, and by experiment to verify them, giving the critical speed in terms of the diameter of the shaft, weights of pulleys, &c., in particular cases applicable to the different conditions under which a shaft works.

In many cases, as might naturally be expected, the "period of whirl" of the shaft is merely its natural period of lateral vibration when in a state of rest. The two periods are coincident in the case of an unloaded shaft (however supported), and for a loaded shaft on which the pulleys are placed in such positions that they rotate—when the shaft is whirling—in planes perpendicular to the original alignment of the shaft. With pulleys placed in any other positions, when the shaft is whirling, there is a righting moment, tending to straighten the shaft, which does not exist when it merely vibrates under the dead weight of the pulleys.

Hence, in an unloaded shaft, the period of whirl coincides with the natural period of lateral vibration; but, generally, in a loaded shaft, the period of whirl is less than the natural period of vibration, to an extent depending on the size and positions of the pulleys.

If, therefore, the period of disturbance (that is, the period of one revolution) be decreased, the shaft runs true until that period approximates to the natural period of vibration of the shaft (assumed at rest) under the given conditions. If the shaft now receive any displacement, however slight, a violent agitation is set up, which will be most marked when the period of disturbance and the whirling period coincide. As the period of disturbance is further decreased, the agitation becomes less, and, at a period of disturbance slightly less than the whirling period of the shaft, the shaft will again run true.

As in the vibration of rods, so in the whirling of shafts, there are a series of periods at which the shaft whirls.

EXPERIMENTAL APPARATUS.

2. The experiments were made in the Whitworth Engineering Laboratory, Owens College, where the essential facilities for obtaining uniform rotation at any

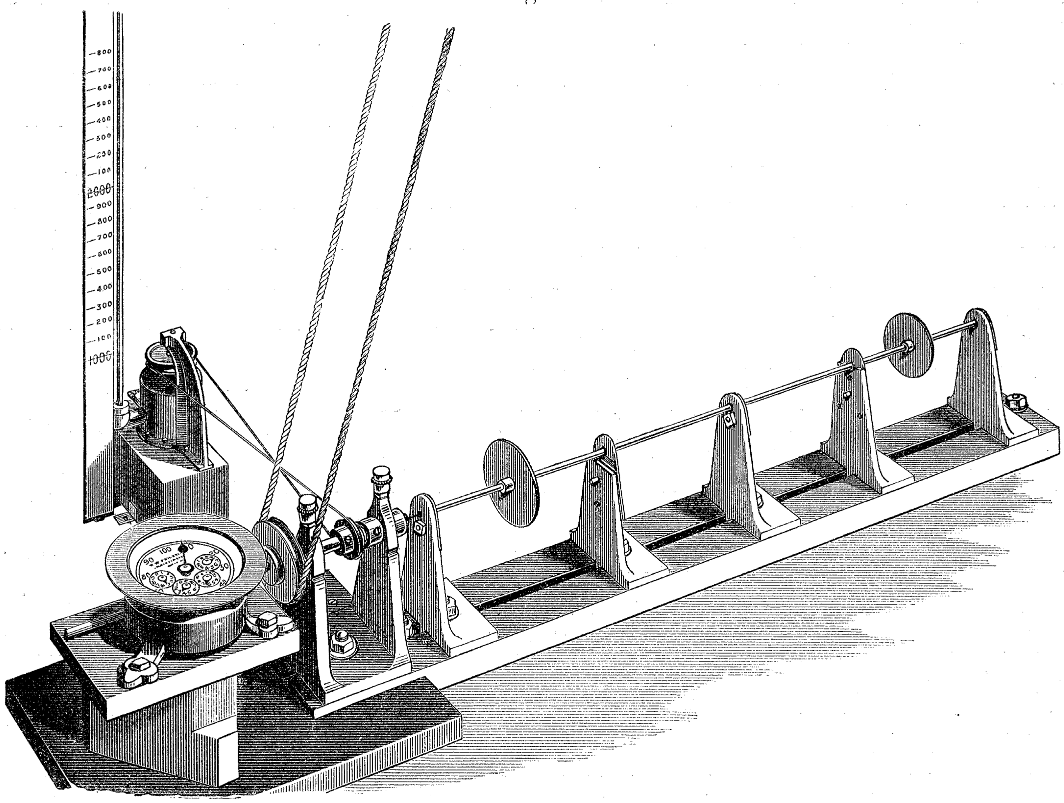
* RANKINE'S 'Machinery and Millwork,' p. 549.

† 'Proc. of Inst. Mech. Engineers,' April, 1883.

speeds were afforded by one of Professor REYNOLDS' quadruple turbines working under a constant head of 113 feet of water.

The essential parts of the apparatus by which the different formulæ were verified consisted of a (see fig. 1) *cast-iron bed plate*, of stiffened channel section, 3 feet 6 inches long and 4 inches wide, with its top and bottom faces planed parallel; a *headstock* which was $7\frac{1}{4}$ inches high, 4 inches wide, and 4 inches long, with its bottom face planed; a *headstock spindle* (which receives the motion), $\frac{1}{2}$ inch diameter, and provided with a shoulder at one end, a loose collar, and two speed pulleys, one

Fig. 1.



for directly receiving the motion, the other for transmitting the motion to a *centrifugal fan indicator*, which approximately indicates the speed of the headstock spindle, at any instant, by the height of a column of liquid forced by the fan up a glass tube. The scale of the indicator was graduated by accurately determining the speeds required to force the liquid up to two or three definite heights, and so obtaining a formula by means of which the heights due to certain speeds can be calculated. The formula so obtained was

$$N = 711h^{\cdot 465}$$

where

N = number of revolutions of headstock spindle per minute,

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and

h = height of liquid, from level of still water, measured in inches.

The scale was graduated for every 100 revolutions per minute.

The *bearings* in which the experimental shaft ran consisted of brass castings of L section with their bottom faces planed. They were bored at exactly the same height as the headstock, and the length of the bearings was about an eighth of an inch. The deflection of the shaft, when whirling, was limited by the use of guard rings, which consisted merely of ordinary bearing castings bored to a slightly larger diameter than the diameter of the shaft.

The motion was transmitted from the shoulder end of the headstock spindle to the experimental shaft by means of a piece of steel wire (about $1\frac{1}{2}$ inch long and 21 B.W.G. diameter), one end of which was soldered into the end of the shaft, the other end being soldered into a piece of brass coned to fit into the headstock spindle. By this means the shaft was subjected to very little constraint.

The headstock spindle was driven from a turbine which was 20 yards away from the experimentalist's bench. The motion was transmitted through 140 feet of quarter-inch cotton rope, the rope ascending vertically from the turbine and descending vertically to the headstock spindle. The admission of water to the turbine was controlled by a hand-wheel close to the apparatus, by which an almost indefinitely fine adjustment of the speed of the turbine could be made from 200 to 2000 revolutions per minute. By having speed-pulleys on the turbine shaft and headstock spindle, a range of speed of from 100 to 10,000 revolutions per minute of the headstock spindle was obtained.

3. In taking the number of revolutions corresponding to any period of whirl, an ordinary counter pushed into the end of the headstock spindle was used. The whirling speed was taken to be at the commencement of whirl, that is to say, at the lowest speed at which the shaft definitely whirled. Readings were taken, in each trial, over a period of from 3 to 5 minutes, the speed (if it varied from some cause) being kept constant by means of the valve regulating wheel. The constancy of speed was shown by the steadiness of the liquid column of the indicator. In making any experiment three trials were made, and the mean of the results taken.

In all cases the theoretical speed was unknown when the actual whirling speed was obtained.

4. The headstock spindle was originally driven by hand. This was accomplished by means of two cast-iron speed pulleys turning on pins bolted to the two ends of a cast-iron bracket, the bracket being bolted to the headstock. By running from a large pulley on the hand-wheel to a small one on the second wheel, and from a large pulley on the second wheel to a small one on the headstock spindle, a very high speed

was attainable. The motion was naturally unsteady, and available only for short periods, whilst an additional observer was required. By driving the shaft from the turbine a practically constant steady speed was obtained, and the increased duration of the trial considerably reduced the personal errors with the counter. Moreover, by an arrangement for regulating the turbine valve at the bench, the action of the shaft could be carefully observed whilst the speed was increased, and so personal errors in determining the precise moment of whirl reduced to a minimum.

5. The EXPERIMENTAL SHAFT was of cast steel. It was 32·18 inches long, and ·2488 inch diameter. The greatest variation in the diameter was $\frac{3}{10,000}$ths of an inch. It was turned by Mr. THOS. FORSTER of the Whitworth Engineering Laboratory, Owens College, Manchester, to whom the author is indebted for the preparation of the greater part of the experimental apparatus.

The shaft weighed 200·2 grms., or ·4414 lb. The weight per foot run was ·1646 lb.

The determination of E (YOUNG'S Modulus), or rather EI (I being the geometrical moment of inertia of the cross-section about a diameter), was accomplished as follows :—The experimental shaft was placed in bearings, 2 feet 8 inches apart, and loaded at the centre. The deflection was measured by means of a micrometer, the distance measured being taken between the top of the shaft and the bottom of a pin fixed in one of the guard castings.

The mean of the results so obtained gives for the

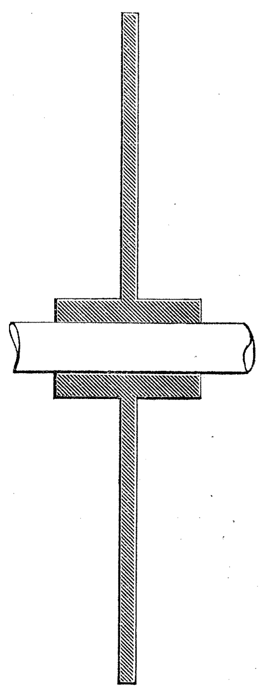
$$\text{Value of EI} = 36{\cdot}554$$
$$\text{,,} \quad \text{E} = 4{,}028{,}200{,}000,$$

the gravitational system of units being employed.

[NOTE.—The value of E expressed in pounds *per square inch* is 27,974,000].

6. The EXPERIMENTAL PULLEYS were of brass and of the section (fig. 2).

Fig. 2.



The moment of inertia taken (for a reason which will appear later) is

$$A - B,$$

where A, B are the mass-moments of inertia about the axis of the shaft, and about a diameter passing through the centre of gravity of the pulley perpendicular to the axis—both moments being expressed in *gravitation units* which, it may be remarked, are the ones adopted throughout the investigation.

The following table gives the dimensions and other necessary information. In it the notation used is as follows :—

W = Weight of pulley.

I′ = Moment of inertia (= A − B).

$k' = \sqrt{\frac{g}{W}(A-B)}$, where $g = 32{\cdot}2$.

E = YOUNG'S Modulus.

I = Geometrical moment of inertia of cross-section of shaft about a diameter.

Name of pulley.	Web.		Nave.		W.	I′.	k'^2.	$\frac{gEI}{W}$.	$\frac{EI}{I'}$.
	Diameter in inches.	Thickness in inches.	Diameter in inches.	Length in inches.					
I.	3·0050	·0497	·46	·622	·1216	·00001207	·003197	9681	3,028,000
II.	3·5134	·0882	·488	·738	·2735	·0000403	·004745	4303	906,700

The pulleys were bored so as to fit the largest part of the shaft, being kept in position on it by rubbing bees-wax on the part of the shaft required, and heating the pulleys sufficiently to melt the wax. On cooling, the wax was sufficient to firmly secure the pulley in its place.

It may be mentioned that PULLEY I. is the model of light pulleys generally used in workshops; whilst PULLEY II. is the model of a 3-feet belt pulley, weighing about 500 lbs. In designing the experimental pulleys, account has, of course, been taken of the different sized shafts on which the actual pulleys run—the pulleys being designed for weight and inertia.

The following are the actual sizes of the pulleys, of which I. and II. are models :—

Model pulley.	Diameter of shaft, in *ins.*	Weight of actual pulley, in *lbs.*	Moment of inertia.
I.	$2\frac{1}{4}$	95	·716
II.	3	490	10·04

CHAPTER II.—GENERAL THEORY, AS GIVEN BY PROFESSOR REYNOLDS.

7. Take the axis of x to be the original alignment of the shaft; and that of y perpendicular to it and revolving with the shaft.

Let

M = bending moment at a distance x from the origin, and let the deflection at this point be y.

C = centrifugal force per unit length of shaft.

I = geometrical moment of inertia of a cross-section of the shaft about a diam.

E = YOUNG's Modulus for the shaft.

ω = angular velocity of shaft.

w = weight of shaft in lbs. per foot run.

W = weight, in lbs., of any pulley which the shaft carries.

I′ = some moment of inertia of the pulley yet to be determined.

Neglecting the dead weight of the shaft, the ordinary equations of the beam give us

$$d^2\mathrm{M}/dx^2 = \mathrm{C} \quad \dots \dots \dots \quad (1),$$

and

$$d^2y/dx^2 = \mathrm{M}/\mathrm{EI} \quad \dots \dots \dots \quad (2),$$

whence

$$\frac{d^4y}{dx^4} = \frac{\mathrm{C}}{\mathrm{EI}} = \frac{1}{\mathrm{EI}}\left(\frac{w}{g}\omega^2 y\right) = m^4 y \quad \dots \dots \dots \quad (3),$$

where

$$m = (w\omega^2/g\mathrm{EI})^{\frac{1}{4}}.$$

Equation (3) holds between every pair of singular points, that is to say, between bearings and pulleys.

At a point of support, the difference of shearing force on the two sides must clearly equal the pressure, that is,

$$d\mathrm{R}/dx - d\mathrm{L}/dx = \mathrm{P} \quad \dots \dots \dots \quad (4),$$

where R and L are the bending moments to the right and left of the support, and P is the pressure on the support.

At a load consisting of a revolving weight W, this equation becomes (neglecting the dead weight of the pulley)

$$d\mathrm{R}/dx - d\mathrm{L}/dx = \mathrm{W}/g\,.\,\omega^2 y \quad \dots \dots \dots \quad (5).$$

A further equation may be obtained by considering the "centrifugal couple" tending to straighten the shaft. The moment of the centrifugal forces about a diametral line in the plane of the pulley and passing through its centre of gravity is $\mathrm{I}'\omega^2.dy/dx$ where $\mathrm{I}' = \mathrm{A} - \mathrm{B}$,

and

A = mass-moment of inertia of pulley about an axis through its centre of gravity perpendicular to its plane, and

B = mass-moment of inertia about a diameter through its centre of gravity perpendicular to the axis of the shaft.

Hence

$$R - L = \omega^2 (A - B)\, dy/dx \quad . \quad . \quad . \quad . \quad (6).$$

8. The solution to equation (3) is well known to be

$$y = A \cosh mx + B \sinh mx + C \cos mx + D \sin mx \quad . \quad . \quad . \quad . \quad (7).$$

The quantities A, B, C, D are absolute constants between any two singular points, but have not necessarily the same values between every pair of singular points.

If undashed symbols refer to those on the left, and dashed constants or symbols to those on the right of a singular point, then since the values of $y = dy/dx$, are continuous, we have, at all singular points, whether points of supports or pulleys,

$$y = y', \quad dy/dx = dy'/dx\,;$$

whence

$$(A - A') \cosh mx + (B - B') \sinh mx + (C - C') \cos mx + (D - D') \sin mx = 0 \quad (8),$$

$$(A - A') \sinh mx + (B - B') \cosh mx - (C - C') \sin mx + (D - D') \cos mx = 0 \quad (9).$$

But, at points of supports, $y = 0$, $y' = 0$; whence

$$A \cosh mx + B \sinh mx + C \cos mx + D \sin mx = 0 \quad . \quad . \quad . \quad (10),$$

$$A' \cosh mx + B' \sinh mx + C' \cos mx + D' \sin mx = 0 \quad . \quad . \quad . \quad (11).$$

Also, since the bending moment is the same on both sides of a point of support, $d^2y/dx^2 = d^2y'/dx^2$, whence

$$(A - A') \cosh mx + (B - B') \sinh mx - (C - C') \cos mx - (D - D') \sin mx = 0 \quad (12).$$

At a singular point, consisting of a concentrated load, we have, from equations (2) and (5),

$$(A - A') \sinh mx + (B - B') \cosh mx + (C - C') \sin mx - (D - D') \cos mx$$
$$= - \frac{W}{m^3 g EI} \omega^2 \{A \cosh mx + B \sinh mx + C \cos mx + D \sin mx\} \quad . \quad . \quad (13),$$

and, from equations (2) and (6),

$$(A - A') \cosh mx + (B - B') \sinh mx - (C - C') \cos mx - (D - D') \sin mx$$
$$= - \frac{\omega^2 I'}{m EI} \{A \sinh mx + B \cosh mx - C \sin mx + D \cos mx\} \quad . \quad . \quad (14).$$

In addition to these equations we shall get equations according to the manner in which the shaft is supported at the ends. If it merely rest on the bearing, so that the bearing exercises no restraint on its direction, the bending moment at that point is zero, that is, $d^2y/dx^2 = 0$, and, therefore,

$$A \cosh mx + B \sinh mx - C \cos mx - D \sin mx = 0 \quad . \quad . \quad . \quad (15).$$

On the other hand, if the bearing be so long that it practically guides the direction of the shaft, in other words, if the shaft be fixed in direction, then we have $dy/dx = 0$, or

$$A \sinh mx + B \cosh mx - C \sin mx + D \cos mx = 0 \quad . \quad . \quad . \quad (16).$$

It will be found that, in every case, equations (8) to (16), inclusive, are sufficient in number to allow of the elimination of the ratios A : B : C : D : A′ : B′ : &c.

The resulting equation will give a relation between the whirling speed, size and weight of the pulleys, diameter of the shaft, &c., that relation depending on the manner in which the shaft is supported and loaded.

The proper value of x has, of course, to be substituted, in the above equations, for any particular singular point.

The values of the constants A, B, C, D at the ends of a shaft are zero.

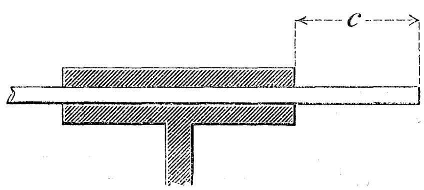
CHAPTER III.—SPECIAL CASES—UNLOADED SHAFTS.

Case I.

9. Overhanging Shaft, length c, fixed in direction at one end.

Thus

Fig. 3.



We have (§ 7, p. 286, equation 3) $d^4y/dx^4 = m^4y$, where $m = (w\omega^2/g\text{EI})^{\frac{1}{4}}$, whence

$$y = A \cosh mx + B \sinh mx + C \cos mx + D \sin mx.$$

Taking the origin at the shoulder, we have, when $x = 0$,

$$y = 0, \quad dy/dx = 0,$$

and, when $x = c$,

$$d^2y/dx^2 = 0, \quad d^3y/dx^3 = 0$$

(shearing force zero). Hence we get

$$A + C = 0 \quad \ldots \ldots \quad (1),$$

$$B + D = 0 \quad \ldots \ldots \quad (2),$$

$$A \cosh mc + B \sinh mc - C \cos mc - D \sin mc = 0 \quad \ldots \quad (3),$$

$$A \sinh mc + B \cosh mc + C \sin mc - D \cos mc = 0 \quad \ldots \quad (4).$$

The elimination of A : B : C : D, from these four equations, leads to either $A = 0$, $B = 0$, $C = 0$, $D = 0$, or to

$$(\cosh mc + \cos mc)^2 - (\sinh mc + \sin mc)(\sinh mc - \sin mc) = 0,$$

i.e.,

$$\cosh mc \cos mc + 1 = 0 \quad \ldots \ldots \quad [A].$$

The least value of mc which satisfies this equation is

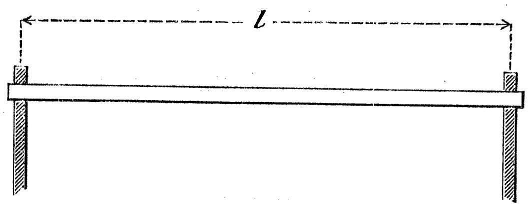
$$mc = 1{\cdot}87001.^*$$

Case II.

10. Shaft, length l, merely resting on a bearing at each end.

Thus—

Fig. 4.



We have (§ 7, p. 286, equation 3) $d^4y/dx^4 = m^4y$, whence

$$y = A \cosh mx + B \sinh mx + C \cos mx + D \sin mx.$$

Taking the origin at the left-hand bearing, we have, when $x = 0$ or l, $y = 0$, $d^2y/dx^2 = 0$, whence

$$A + C = 0 \quad \ldots \ldots \quad (1),$$

$$A - C = 0 \quad \ldots \ldots \quad (2),$$

$$A \cosh ml + B \sinh ml + C \cos ml + D \sin ml = 0 \quad \ldots \quad (3),$$

$$A \cosh ml + B \sinh ml - C \cos ml - D \sin ml = 0 \quad \ldots \quad (4).$$

* Poisson, 'Traité de Mécanique,' vol. 2, § 528.

The elimination of A : B : C : D from these equations gives either A = 0, B = 0, C = 0, D = 0, or sin $ml = 0$, *i.e.*,

$$ml = 3{\cdot}1416.$$

11. *Experimental Results.*—For a description of the manner in which the experiments were made, see § 3, p. 283.

The following are the mean results—the percentage errors being considered positive or negative according as the observed is greater or less than the calculated speed.

Number of experiment.	Date.	Conditions.	Observed speed.	Calculated speed.	Percentage error being $100 \times \frac{\text{observed-calculated}}{\text{observed}}$.
2	June 22, 1892	Free span of 2′ 8″	1119	1121	− ·2
1	" "	" " 2′ 6″	1293	1275	+ ·4

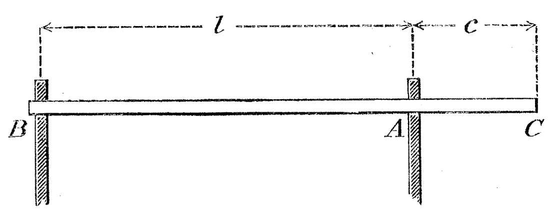
The experiments show that, in the simple case of a shaft resting on two bearings in which the conditions required by the theory can be very closely approximated to in practice, there is no appreciable difference between the observed and calculated speeds.

Case III.

12. SHAFT SUPPORTED ON BEARINGS l FEET APART AND OVERHANGING TO A LENGTH c ON ONE SIDE.

Thus—

Fig. 5.



Taking the origin at A, we have from B to A,

$$y = \text{A} \cosh mx + \text{B} \sinh mx + \text{C} \cos mx + \text{D} \sin mx,$$

and, from A to C,

$$y' = \text{A}' \cosh mx + \text{B}' \sinh mx + \text{C}' \cos mx + \text{D}' \sin mx$$

(§ 8, p. 287, equation 7).

When $x = -l$,

$$y = 0, \qquad d^2y/dx^2 = 0,$$

and when $x = 0$,

$$y = 0, \qquad y' = 0, \qquad dy/dx = dy'/dx, \qquad d^2y/dx^2 = d^2y'/dx^2.$$

Also, when $x = c$,

$$d^2y'/dx^2 = 0, \qquad d^3y'/dx^3 = 0 \text{ (shearing force zero).}$$

Hence, we get

$$\text{A} \cosh ml - \text{B} \sinh ml + \text{C} \cos ml - \text{D} \sin ml = 0 \quad . \quad . \quad . \quad . \quad (1),$$

$$\text{A} \cosh ml - \text{B} \sinh ml - \text{C} \cos ml + \text{D} \sin ml = 0 \quad . \quad . \quad . \quad . \quad (2),$$

$$\text{A} + \text{C} = 0 \quad . \quad . \quad . \quad . \quad (3),$$

$$\text{A}' + \text{C}' = 0 \quad . \quad . \quad . \quad . \quad (4),$$

$$(\text{B} - \text{B}') + (\text{D} - \text{D}') = 0 \quad . \quad . \quad . \quad . \quad (5),$$

$$(\text{A} - \text{A}') - (\text{C} - \text{C}') = 0 \quad . \quad . \quad . \quad . \quad (6),$$

$$\text{A}' \cosh mc + \text{B}' \sinh mc - \text{C}' \cos mc - \text{D}' \sin mc = 0 \quad . \quad . \quad . \quad . \quad (7),$$

$$\text{A}' \sinh mc + \text{B}' \cosh mc + \text{C}' \sin mc - \text{D}' \cos mc = 0 \quad . \quad . \quad . \quad . \quad (8).$$

The elimination of A : B : C : D : A′ : B′ : C′ : D′ from these equations leads to the result

$$(\cosh ml \sin ml - \sinh ml \cos ml) \times (\cosh mc \sin mc - \sinh mc \cos mc)$$
$$- 2 \sinh ml \sin ml\,(1 + \cosh mc \cos mc) = 0 \quad . \quad . \quad . \quad . \quad [\text{A}].$$

If $l = 0$, by dividing throughout by $\sinh ml \sin ml$, the equation reduces to

$$1 + \cosh mc \cos mc = 0$$

the equation already obtained for an overhanging shaft fixed in direction at one end (Case 1, § 9, p. 289).

If $c = 0$, the equation [A] reduces to $\sinh ml \sin ml = 0$, *i.e.*, $\sin ml = 0$, the equation already obtained for a shaft resting freely on a bearing at each end (Case II, § 10, p. 290).

The general solution to equation [A] is best obtained by assuming $c = al$, where a is less than unity, and expanding each term in ascending powers of ml. In this manner we get, to a sufficient degree of approximation, the equation

$$(ml)^8 \left\{\frac{a^4}{270} + \frac{2a^3}{945}\right\} - (ml)^4 \left\{\frac{a^4}{3} + \frac{4a^3}{9} + \frac{2}{45}\right\} + 4 = 0.$$

From this equation the following results—giving the values of ml for different values of a—have been obtained.

Ratio a.	Value of ml.
Unity	1·506
Three-quarters	1·902
One-half	2·507
One-third	2·905
One-quarter	3·009
One-fifth	3·044
One-sixth	3·060
One-seventh	3·069
One-eighth	3·071
One-ninth	3·073
One-tenth	3·078
Very small	3·080

If we assume $ml = \text{A}$, then the number of revolutions will be a maximum for a given length $(l + c)$ of shafting, when $\text{A}(1 + a)$ is a maximum. From the above results the speed will be a maximum when the ratio (a) is one-third.

Hence, for a shaft of given length running on two bearings, one being placed at the end, the best position for the other bearing is such that it divides the length of the shafting in the proportion of 1 : 3.

In all cases that occur in practice the overhanging portion is small compared to the span. Hence, we may say that if a shaft, span l, overhang a distance less than one-fifth the span, then $ml = 3·078$.

13. *Experimental Results.*

The following are the mean results, the calculated speeds being obtained according to the formulæ in the preceding article (p. 292), when the particular value of c/l is taken.

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Number of Experiment.	Date.	Conditions.			Observed speed.	Calculated speed.	Percentage error being $100 \times \frac{\text{observed-calculated}}{\text{observed}}$.
		Ratio $= \frac{c}{l}$.	Span in inches (l).	Overhanging portion in inches (c).			
	1892						
24	Oct. 19	$\frac{1}{10}$	29·10	2·91	1309	1301	+ ·6
25	,, 19	$\frac{1}{9}$	28·80	3·20	1355	1324	+ 2·3
27	,, 20	$\frac{1}{8}$	28·44	3·56	1372	1356	+ 1·2
26	,, 19	$\frac{1}{7}$	28·00	4·00	1435	1397	+ 2·6
28	,, 20	$\frac{1}{6}$	27·42	4·57	1456	1448	+ ·5
29	,, 20	$\frac{1}{5}$	26·66	5·33	1472	1516	− 3·0
30	,, 20	$\frac{1}{4}$	25·60	6·40	1545	1603	− 2·9
31	,, 20	$\frac{1}{3}$	24·00	8·00	1606	1704	− 6·1
32	,, 20	$\frac{1}{2}$	21·33	10·66	1558	1606	− 3·1
33	,, 20	$\frac{3}{4}$	18·30	13·70	1201	1256	− 4·6
34	,, 20	1	16·00	16·00	1002	1031	− 2·9

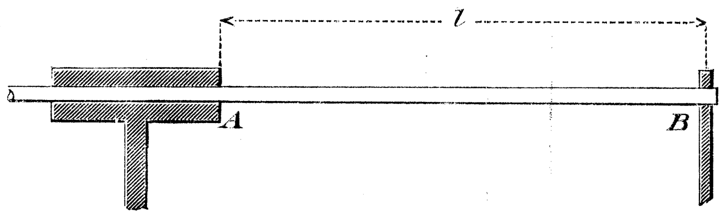
These results show that the calculated speeds are less than the observed speeds provided α (that is c/l) be less than one-fifth (which is always the case in practice), and in excess for greater values of c/l. In only two cases is the percentage error greater than 3 per cent., thus amply verifying the theory. The maximum observed speed is when $c/l = 1/3$, a result which has been shown to follow immediately from the equations.

Case IV.

14. Shaft, length l resting freely on a support at one end and fixed in direction at the other.

Thus—

Fig. 6.



We have (§ 8, p. 287, equation 7)

$$y = \text{A} \cosh mx + \text{B} \sinh mx + \text{C} \cos mx + \text{D} \sin mx.$$

Taking the origin at A, we have, when $x = 0$,

$$y = 0, \qquad dy/dx = 0;$$

and when $x = l$,

$$y = 0, \qquad d^2y/dx^2 = 0.$$

Hence,

$$A + C = 0 \quad \ldots \quad (1),$$

$$B + D = 0 \quad \ldots \quad (2),$$

$$A \cosh ml + B \sinh ml + C \cos ml + D \sin ml = 0 \quad \ldots \quad (3),$$

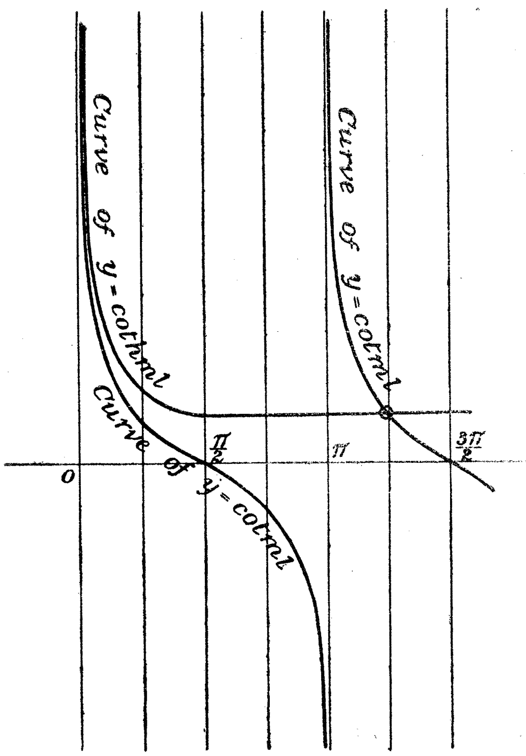
$$A \cosh ml + B \sinh ml - C \cos ml - D \sin ml = 0 \quad \ldots \quad (4).$$

The elimination of A : B : C : D from these equations leads to

$$\coth ml = \cot ml.$$

To solve this equation, draw the curves of coth ml and cot ml. The points of intersection of $y = \coth ml$ with $y = \cot ml$ will give values of ml which satisfy the equation $\coth ml = \cot ml$.

Fig. 7.



From the diagram, it will be seen that the first value of ml is less than $\pi + \frac{1}{4}\pi$ by a small quantity. It may be shown that, to a sufficient degree of approximation,

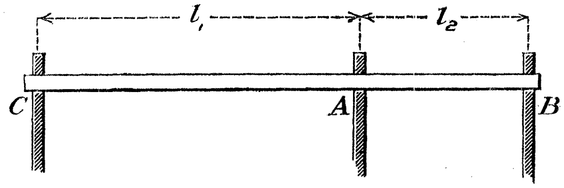
$$ml = 3{\cdot}9266.$$

Case V.

15. Shaft supported on three supports, l_1 and l_2 feet apart respectively, a support being at each end.

Thus—

Fig. 8.



Take the origin at A, and let dashed letters refer to the right of A, and undashed letters to the left. Then we have (§ 8, p. 287, equation 7) from C to A,

$$y = \text{A} \cosh mx + \text{B} \sinh mx + \text{C} \cos mx + \text{D} \sin mx,$$

and from A to B,

$$y' = \text{A}' \cosh mx + \text{B}' \sinh mx + \text{C}' \cos mx + \text{D}' \sin mx.$$

When $x = 0$

$$y = 0, \qquad y' = 0, \qquad dy/dx = dy'/dx, \qquad d^2y/dx^2 = d^2y'/dx^2\,;$$

when $x = -l_1$

$$y = 0, \qquad d^2y/dx^2 = 0\,;$$

when $x = l_2$

$$y' = 0, \qquad d^2y'/dx^2 = 0.$$

Hence we get

$$\text{A} + \text{C} = 0 \quad \ldots \quad (1),$$

$$\text{A}' + \text{C}' = 0 \quad \ldots \quad (2),$$

$$(\text{B} - \text{B}') + (\text{D} - \text{D}') = 0 \quad \ldots \quad (3),$$

$$(\text{A} - \text{A}') - (\text{C} - \text{C}') = 0 \quad \ldots \quad (4),$$

$$\text{A} \cosh ml_1 - \text{B} \sinh ml_1 + \text{C} \cos ml_1 - \text{D} \sin ml_1 = 0 \quad \ldots \quad (5),$$

$$\text{A} \cosh ml_1 - \text{B} \sinh ml_1 - \text{C} \cos ml_1 + \text{D} \sin ml_1 = 0 \quad \ldots \quad (6),$$

$$\text{A}' \cosh ml_2 + \text{B}' \sinh ml_2 + \text{C}' \cos ml_2 + \text{D}' \sin ml_2 = 0 \quad \ldots \quad (7),$$

$$\text{A}' \cosh ml_2 + \text{B}' \sinh ml_2 - \text{C}' \cos ml_2 - \text{D}' \sin ml_2 = 0 \quad \ldots \quad (8).$$

By eliminating A : B : C : D : A′ : B′ : C′ : D′ from these equations we obtain the results, that either A = 0 or

$$\coth ml_1 + \coth ml_2 = \cot ml_1 + \cot ml_2.$$

First consider the solution A = 0.

It follows that B, B′, C, C′, A′ are all zero, and that

$$D = D', \qquad D' \sin ml_2 = 0, \qquad D \sin ml_1 = 0.$$

Hence, in addition to the solution

$$\coth ml_1 + \coth ml_2 = \cot ml_1 + \cot ml_2,$$

the equations (1)–(8) are satisfied when

$$\left.\begin{aligned} ml_1 &= a\pi \\ ml_2 &= b\pi \end{aligned}\right\}\text{simultaneously,}$$

a and b being integers. Hence, if b be a multiple of a, that is, if l_2 be a multiple of l_1, one speed of whirl is clearly that of the shorter span when the longer span is neglected—a result, of course, identical with the vibration of strings in segments.

Secondly, consider the solution

$$\coth ml_1 + \coth ml_2 = \cot ml_1 + \cot ml_2.$$

If $l_1 = l_2$ or $l_2 = 0$, we get

$$\coth ml_1 = \cot ml_1.$$

The physical interpretation of this equation is that the shaft in the one case is horizontal at the middle bearing, and in the other at the end bearing. In other words we get Case IV., § 14, p. 294, which has already been solved.

[It should be noticed that the case when $l_1 = l_2$ comes under the first solution.]

The solution to the general equation

$$\coth ml_1 + \coth ml_2 = \cot ml_1 + \cot ml_2$$

may be performed by putting

$$ml_2 = a \,.\, ml_1,$$

where a is the ratio of the spans, being always less than unity. By expanding we obtain the equation

$$(ml_1^8)\{38a^8 + 23a^7 - 488a^6 - 562a^5 + 76a^4 - 562a^3 - 488a^2 + 23a + 30\}$$
$$- 31680\,(ml)^4\{3a^4 + 4a^3 - 4a^2 + 4a + 3\} + 19958400 = 0.$$

The following are the results obtained from this equation, the value of ml having been calculated for different values of a.

Value of $a = l_2/l_1$.	Value of ml_1.
Very small	3·9003
$\frac{1}{10}$, $\frac{1}{9}$, or $\frac{1}{8}$	3·7620
$\frac{1}{7}$, $\frac{1}{6}$, or $\frac{1}{5}$	3·6480
$\frac{1}{4}$	3·6056
$\frac{1}{3}$	3·5101
$\frac{1}{2}$ to $\frac{3}{4}$	3·3282
$\frac{3}{4}$ to 1	3·1416

The formula is not sufficiently approximate if $a > \frac{1}{2}$.

When a is very small ($< \frac{1}{10}$) the result closely approximates to the result obtained for a shaft working in a sleeve at one end, viz. :

$$ml = 3·9266. \quad (\S\ 14,\ \text{p. } 294.)$$

16. *Experimental Results.*

The following are the mean results, the calculated speeds being obtained according to the formulæ in the preceding article when the particular value of l_1/l_2 is taken.

Number of Experiment.	Date.	Conditions. Ratio l_2/l_1.	Shorter span (l_2) in inches.	Longer span (l_1) in inches.	Observed speed.	Calculated speed.	Percentage error being $100 \times \frac{\text{observed} - \text{calculated}}{\text{observed speed.}}$
	1892.						
36	Oct. 21	$\frac{1}{10}$	2·91	29·10	1942	1943	·0
37	,, 21	$\frac{1}{7}$	4·00	28·00	2051	1974	+ 3·7
42	,, 22	$\frac{1}{6}$	4·57	27·42	2035	2058	− 1·1
38	,, 21	$\frac{1}{4}$	6·40	25·60	2251	2307	− 2·4
41	,, 22	$\frac{1}{3}$	8·00	24·00	2500	2487	+ ·5
39	,, 22	$\frac{1}{2}$	10·66	21·33	3020	2830	+ 6·2
40	,, 22	$\frac{3}{4}$	13·70	18·20	3873	3889	− ·4
3	May 9	1	15·00	15·00	5137	5100	+ ·7
56	Nov. 5	1	16·00	16·00	4390	4484	− 2·1

It will be noticed that in some of these experiments the observed speed is greater, and in others less, than the calculated speed.

With the exception of Experiment 39, the experiments amply verify the theory. From Experiment 39 it would appear that when the ratio of the spans is one-half, the calculated speed is less than the observed speed. It, therefore, errs on the right side.

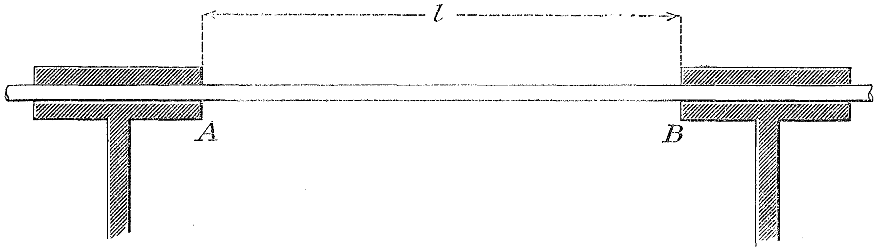
It appears from Experiment 40, that the same formula as for $a = \frac{1}{2}$ holds, to a sufficient degree of approximation, until the ratio of the spans is equal to $\frac{3}{4}$.

Case VI.

17. Shaft, length l, fixed in direction at each end.

Thus—

Fig. 9.



Taking the origin at A, we have (§ 8, p. 287, equation 7)

$$y = \text{A} \cosh mx + \text{B} \sinh mx + \text{C} \cos mx + \text{D} \sin mx$$

when $x = 0$, or l,

$$y = 0,$$

$$dy/dx = 0\,;$$

whence

$$\text{A} + \text{C} = 0 \quad . \quad . \quad . \quad (1),$$

$$\text{B} + \text{D} = 0 \quad . \quad . \quad . \quad (2),$$

$$\text{A} \cosh ml + \text{B} \sinh ml + \text{C} \cos ml + \text{D} \sin ml = 0 \quad . \quad . \quad . \quad (3),$$

$$\text{A} \sinh ml + \text{B} \cosh ml - \text{C} \sin ml + \text{D} \cos ml = 0 \quad . \quad . \quad . \quad (4).$$

The elimination of A : B : C : D from these equations leads to

$$\cosh ml \cos ml - 1 = 0.$$

The least value of ml which satisfies this equation is

$$ml = 4{\cdot}74503.^{*}$$

* Poisson, 'Traité de Mécanique,' vol. 2, § 528.

CHAPTER IV.—SPECIAL CASES—LOADED SHAFTS.

18. In considering shafts loaded with pulleys two methods may be adopted.

First. The period of whirl may be calculated taking both the shaft and pulleys into account together.

Second. The period of whirl may be first calculated for the shaft, neglecting the pulleys, and then for the pulleys, neglecting the shaft. By means of an approximate formula, the period of whirl, taking both shaft and pulleys into account, may be calculated from the separately calculated periods of whirl.

FIRST METHOD OF SOLUTION.

Investigation shows that the first method leads to equations which are not solvable, so as to give results in a form convenient for actual use.

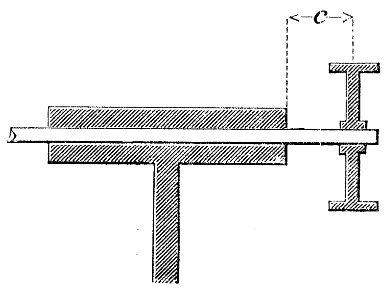
The following two simple cases will illustrate this.

Case VII.

19. OVERHANGING SHAFT, LENGTH c, FIXED IN DIRECTION AT ONE END, AND LOADED WITH A PULLEY, WEIGHT W AND MOMENT OF INERTIA I′, AT ITS FREE END, THE COMBINED EFFECTS OF BOTH SHAFT AND PULLEY BEING TAKEN INTO ACCOUNT.

Thus—

Fig. 10.



we have (§ 7, p. 286, equation 3) for every point between the bearing and the pulley,

$$d^4y/dx^4 = m^4y, \qquad \text{where } m = (w\omega^2/g\text{EI})^{\frac{1}{4}};$$

whence

$$y = \text{A}\cosh mx + \text{B}\sinh mx + \text{C}\cos mx + \text{D}\sin mx.$$

Taking the origin at the shoulder, we have at a singular point consisting of a concentrated load,

$$d\text{R}/dx - d\text{L}/dx = \text{W}/g\,.\,\omega^2y;\ (\S\ 7,\ \text{p. } 286,\ \text{equation } 5).$$

whence, when $x = c$, since $\mathrm{R} = 0$, we get

$$\mathrm{A}\left\{\sinh mc + \frac{\mathrm{W}\omega^2}{m^3 g \mathrm{EI}}\cosh mc\right\} + \mathrm{B}\left\{\cosh mc + \frac{\mathrm{W}\omega^2}{m^3 g\mathrm{EI}}\sinh mc\right\}$$
$$+ \mathrm{C}\left\{\sin mc + \frac{\mathrm{W}\omega^2}{m^3 g\mathrm{EI}}\cos mc\right\} - \mathrm{D}\left\{\cos mc - \frac{\mathrm{W}\omega^2}{m^3 g\mathrm{EI}}\sin mc\right\} = 0 \quad . \quad (1).$$

Again, from equation (6), p. 287, we have

$$\mathrm{R} - \mathrm{L} = \omega^2 \mathrm{I}' \, dy/dx\,;$$

wherefore, when $x = c$,

$$\mathrm{A}\left\{\cosh mc + \frac{\omega^2\mathrm{I}'}{m\mathrm{EI}}\sinh mc\right\} + \mathrm{B}\left\{\sinh mc + \frac{\omega^2\mathrm{I}'}{m\mathrm{EI}}\cosh mc\right\}$$
$$- \mathrm{C}\left\{\cos mc + \frac{\omega^2\mathrm{I}'}{m\mathrm{EI}}\sin mc\right\} - \mathrm{D}\left\{\sin mc - \frac{\omega^2\mathrm{I}'}{m\mathrm{EI}}\cos mc\right\} = 0 \quad . \quad (2).$$

Again, when $x = 0$,

$$y = 0, \qquad dy/dx = 0\,;$$

whence

$$\mathrm{A} + \mathrm{C} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

$$\mathrm{B} + \mathrm{D} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

The elimination of A : B : C : D from the four marked equations leads to

$$\cosh mc\left\{\cos mc\left(1 + \frac{\mathrm{W}\omega^4\mathrm{I}'}{m^4 g\mathrm{E}^2\mathrm{I}^2}\right) + \sin mc\left(\frac{\omega^2\mathrm{I}'}{m\mathrm{EI}} - \frac{\mathrm{W}\omega^2}{m^3 g\mathrm{EI}}\right)\right\}$$
$$+ \sinh mc\cos mc\left\{\frac{\mathrm{W}\omega^2}{m^3 g\mathrm{EI}} + \frac{\omega^2\mathrm{I}'}{m\mathrm{EI}}\right\} + \left\{1 - \frac{\mathrm{W}\omega^4\mathrm{I}'}{m^4 g\mathrm{E}^2\mathrm{I}^2}\right\} = 0 \quad . \quad [\mathrm{A}].$$

If we assume the pulley to be removed, that is, if we put

$$\mathrm{W} = 0, \qquad \mathrm{I}' = 0,$$

in equation [A], we obtain

$$\cosh mc\cos mc + 1 = 0,$$

the same as that obtained in Case I., p. 288.

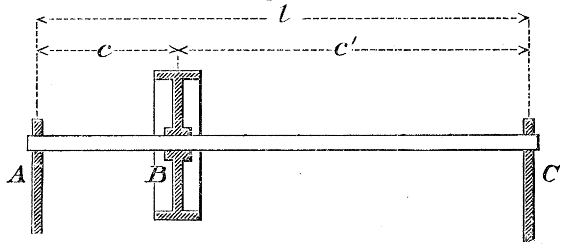
The equation [A] can only be solved by assuming some relation between the coefficients; in other words, we cannot obtain a general solution which could be readily applied in any actual case.

Case VIII.

20. Shaft length l, merely resting on a support at each end, and loaded with a pulley, weight W and moment of inertia I′ at distances c, c' from the supports.

Thus—

Fig. 11.



Taking the origin at A, we have (§ 8, p. 287, equation 7) from A to B,

$$y = \mathrm{A} \cosh mx + \mathrm{B} \sinh mx + \mathrm{C} \cos mx + \mathrm{D} \sin mx,$$

and from B to C,

$$y' = \mathrm{A}' \cosh mx + \mathrm{B}' \sinh mx + \mathrm{C}' \cos mx + \mathrm{D}' \sin mx,$$

where dashed letters refer to the right, and undashed letters to the left of the pulley. At the pulley, when $x = c$, we have (§ 8, p. 287, equation 13)

$$(\mathrm{A} - \mathrm{A}') \sinh mc + (\mathrm{B} - \mathrm{B}') \cosh mc + (\mathrm{C} - \mathrm{C}') \sin mc - (\mathrm{D} - \mathrm{D}') \cos mc$$
$$= - \frac{\mathrm{W}\omega^2}{m^3 g \mathrm{EI}} (\mathrm{A} \cosh mc + \mathrm{B} \sinh mc + \mathrm{C} \cos mc + \mathrm{D} \sin mc) \quad . \quad . \quad . \quad (1).$$

Again, from equation 14, p. 287, we have

$$(\mathrm{A} - \mathrm{A}') \cosh mc + (\mathrm{B} - \mathrm{B}') \sinh mc - (\mathrm{C} - \mathrm{C}') \cos mc - (\mathrm{D} - \mathrm{D}') \sin mc$$
$$= - \frac{\omega^2 \mathrm{I}'}{m \mathrm{EI}} (\mathrm{A} \sinh mc + \mathrm{B} \cosh mc - \mathrm{C} \sin mc + \mathrm{D} \cos mc) \quad . \quad . \quad . \quad . \quad (2),$$

when

$$x = c, \qquad y = y', \qquad dy/dx = dy'/dx,$$

whence

$$(\mathrm{A} - \mathrm{A}') \cosh mc + (\mathrm{B} - \mathrm{B}') \sinh mc + (\mathrm{C} - \mathrm{C}') \cos mc + (\mathrm{D} - \mathrm{D}') \sin mc = 0 \quad (3).$$

$$(\mathrm{A} - \mathrm{A}') \sinh mc + (\mathrm{B} - \mathrm{B}') \cosh mc - (\mathrm{C} - \mathrm{C}') \sin mc + (\mathrm{D} - \mathrm{D}') \cos mc = 0 \quad (4).$$

Again, when $x = 0$, or l,

$$y = 0,$$

$$d^2y/dx^2 = 0,$$

whence

$$A + C = 0 \quad \ldots \ldots \quad (5),$$

$$A - C = 0 \quad \ldots \ldots \quad (6),$$

$$A' \cosh ml + B' \sinh ml + C' \cos ml + D' \sin ml = 0 \quad \ldots \quad (7),$$

$$A' \cosh ml + B' \sinh ml - C' \cos ml - D' \sin ml = 0 \quad \ldots \quad (8).$$

The elimination of A : B : C : D : A′ : B′ : C′ : D′ from these eight equations leads to

$$\begin{aligned} &2 \sinh ml\,(\alpha \sin mc \sin mc' + \beta \cos mc \cos mc') \\ &\quad - 2 \sin ml\,(\alpha \sinh mc \sinh mc' + \beta \cosh mc \cosh mc') - 4 \sin ml \sinh ml \\ &\quad + \alpha\beta\,\{(\cos mc \cos mc' \sinh mc \sinh mc' + \sin mc \sin mc' \cosh mc \cosh mc') \\ &\quad - (\sin mc \cos mc' \cosh mc \sinh mc' + \cos mc \sin mc' \sinh mc \cosh mc')\} = 0 \quad [A], \end{aligned}$$

where

$$\alpha = W\omega^2/m^3 g EI, \qquad \beta = \omega^2 I'/mEI.$$

Equation [A] is, of course, symmetrical with respect to c, c'.

If we imagine the pulley to be removed (by putting $W = 0$ and $I' = 0$) the equation A reduces to

$$\sin ml \sinh ml = 0,$$

i.e.,

$$ml = \pi,$$

a result already obtained in Case II., p. 290.

As in Case VII., § 19, we cannot obtain a general solution to [A], which could be readily applied in any actual case.

Second Method of Solution.

21. The formulæ obtained by considering the effect of the pulleys and the shaft combined have thus been shown, even in simple cases, to be absolutely useless for practical purposes.

By the second method of solution the whirling speed of the pulley neglecting the shaft is first obtained. The general theory (Chapter II.) will have, therefore, to be slightly modified.

Since

$$w = 0,$$

equation 3, of § 7, becomes

$$d^4y/dx^4 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

and, therefore,

$$y = \frac{A}{6}x^3 + \frac{B}{2}x^2 + Cx + D \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

If, as before (§ 8), undashed symbols refer to the symbols or constants on the left, and dashed symbols to those on the right of a singular point, then (as in Chapter II., equations 7–16) we shall have precisely the same differential equations holding at the specified singular points, the only difference being that when those differential equations are integrated, the forms of the resulting equations are altered from a trigonometrical (in Chapter II.) to an algebraic form in the present case.

22. It is now proposed to investigate some of the cases, commonly occurring in practice, according to the second method of solution. Whatever be the manner in which the shaft is supported, the effect of the shaft is neglected, and the shaft supposed to be loaded with one pulley only.

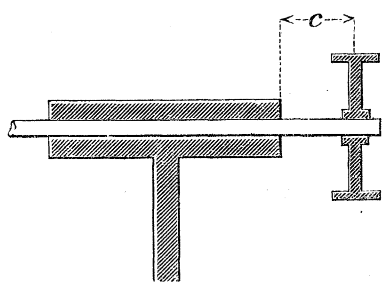
The effect of the shaft, and of more than one pulley, will be considered in §§ 59–62.

Case IX.

23. Overhanging shaft, length c, fixed in direction at one end, and loaded with a pulley, weight W, and moment of inertia, I′ at its end.

Thus—

Fig. 12.



We have (§ 21, equation 2),

$$y = \frac{A}{6}x^3 + \frac{B}{2}x^2 + Cx + D.$$

Taking the origin at the shoulders, we have, when $x = 0$,

$$y = 0, \qquad dy/dx = 0,$$

whence

$$D = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

$$C = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

When $x = c$, that is, at the pulley, we have

$$\left.\begin{aligned} R - L &= \omega^2 I' \frac{dy}{dx} \\ \frac{dR}{dx} - \frac{dL}{dx} &= \frac{W}{g}\,\omega^2 y \end{aligned}\right\} \quad \begin{aligned} &(\S\ 7,\ \text{equation } 6), \\ &(\S\ 7,\ \text{equation } 5). \end{aligned}$$

whence

$$d^2y/dx^2 = -\,\omega^2 I'/EI \,.\, dy/dx,$$

and

$$d^3y/dx^3 = -\,W/gEI \,.\, \omega^2 y,$$

or

$$Ac + B = -\frac{\omega^2 I'}{EI}\left\{\frac{A}{2}c^2 + Bc + C\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

and

$$A = -\frac{W}{gEI}\,\omega^2\left\{\frac{A}{6}c^3 + \frac{B}{2}c^2 + Cc + D\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Let

$$\alpha = W\omega^2/gEI, \qquad \beta = I'\omega^2/EI,$$

so that $\beta = \alpha k^2$ where $k = \sqrt{(gI'/W)}$, I' having the value assigned to it in § 7.

The elimination of A : B : C : D from the four equations marked leads to

$$\tfrac{1}{12}\alpha\beta c^4 + \tfrac{1}{3}\alpha c^3 - (\beta c + 1) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [A],$$

whence

$$\omega^2 = \frac{gEI}{Wc^3}\left\{\left(6 - \frac{2c^2}{k^2}\right) \pm \sqrt{\left(6 - \frac{2c^2}{k^2}\right)^2 + \frac{12c^2}{k^2}}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad [B].$$

24. Equation [A] may be put in the form

$$k^2 = \frac{3 - \alpha c^3}{\alpha c^3 - 12} \cdot \frac{4}{\alpha c} \cdot$$

If αc^3 be < 3 or > 12, k^2 is negative, and therefore the equations do not hold. Hence, for whirling to be at all possible, αc^3 must be > 3 and < 12; that is, $\omega^2 . Wc^3/gEI$ must lie between 3 and 12.

The speeds which these values give for any value of c may be termed the *inferior and superior limits of the speed.*

The values of k corresponding to these limits are zero and infinity. In other words, if the shaft whirl at a speed which satisfies

$$\omega^2 . Wc^3/gEI = 3 \text{ or } 12,$$

the effect of the inertia of the pulley is either zero or infinity. In the first case we should have zero righting moment, and in the second, an infinite righting moment. In other words, in the one case there would be no tendency to make the pulley deviate from its natural plane of rotation, and in the other, any such tendency would be met by an infinite moment tending immediately to right it. In either case, therefore—assuming whirling to take place at the speeds given by the limiting values of αc^3—it would whirl in such a manner that the pulley still rotates in a plane perpendicular to the original alignment of the shaft.

In fact, *the period of whirling, corresponding to the inferior limit of the speed, is identical with the natural period of vibration of the light shaft under the given conditions.*

This may be easily proved independently.*

The superior limit is double the inferior limit.

The inferior limit may be taken as a first approximation to the period of whirl.

25. Referring to equation [B], § 23, by giving c/k different values likely to be met with in practice, we get, for each value of c/k, a relation between ω, the angular velocity of whirl, and c, the overhanging portion. Knowing, therefore, the particular value of c, the value of ω may be readily calculated.

The following are the results obtained in this manner from equation [B]:—

* This may be seen as follows:—

If W be the weight of the pulley, and ϵ the force necessary to deflect it one foot, then t (the time of lateral vibration) is $2\pi\sqrt{(W/g\epsilon)}$. To get ϵ, if P be the load acting at a distance c from the shoulder, as in fig. 12, M the bending moment at a distance x from the shoulder, then

$$M = Px,$$
$$d^2y/dx^2 = M/EI = Px/EI,$$

where E and I have the same meaning as in the text.

Hence,

$$y = \frac{Px^3}{6EI} + Ax + B,$$

where A and B are constants of integration. When $x = 0$, $y = 0$, and $dy/dx = 0$; whence $B = 0$, $A = 0$, and

$$y = Px^3/6EI.$$

The deflection, therefore, at the weight is $Pc^3/6EI$, and $P = \epsilon$ when this is unity. Hence $\epsilon = 6EI/c^3$ and, therefore,

$$t = \text{natural period of lateral vibration}$$
$$= 2\pi\sqrt{(Wc^3/6gEI)}.$$

Whence

$$\omega = 2\pi/t = \sqrt{(6gEI/Wc^3)} = 1{\cdot}732\sqrt{(gEI/Wc^3)}.$$

Values of θ in the equation $\omega = \theta\sqrt{(g\text{EI}/\text{W}c^3)}$, c being the Distance of the Pulley from the Shoulder.

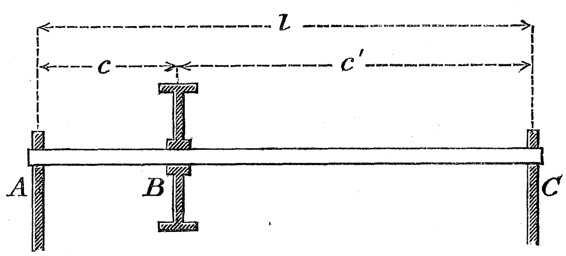
Value of c/k.	Value of θ.
Small (superior limit)	3·464
·25	3·437
·50	3·356
·75	3·225
1·00	3·048
1·25	2·841
1·50	2·628
1·75	2·437
2·00	2·282
Large (inferior limit)	1·732

Case X.

26. SHAFT, LENGTH l, MERELY RESTING ON A SUPPORT AT EACH END AND LOADED WITH A PULLEY, WEIGHT W AND MOMENT OF INERTIA I′, AT DISTANCES c, c' FROM THE SUPPORTS.

Thus—

Fig. 13.



Let the origin be taken at the left-hand bearing.

We have (§ 21, equation 2)

$$y = \frac{\text{A}}{6}x^3 + \frac{\text{B}}{2}x^2 + \text{C}x + \text{D}$$

between A and B, and

$$y' = \frac{\text{A}'}{6}x^3 + \frac{\text{B}'}{2}x^2 + \text{C}'x + \text{D}'$$

between B and C.

When $x = 0$,

$$y = 0, \qquad d^2y/dx^2 = 0$$

Therefore

$$D = 0 \quad \ldots \ldots \ldots \ldots \quad (1),$$

$$B = 0 \quad \ldots \ldots \ldots \ldots \quad (2).$$

When

$$x = l, \qquad y' = 0, \qquad d^2y'/dx^2 = 0\,;$$

therefore

$$\frac{A'}{6}l^3 + \frac{B'}{2}l^2 + C'l + D' = 0 \quad \ldots \ldots \ldots \quad (3),$$

$$A'l + B' = 0 \quad \ldots \ldots \ldots \ldots \quad (4).$$

At the pulley, when $x = c$,

$$y = y', \qquad dy/dx = dy'/dx\,;$$

therefore

$$\frac{A - A'}{6}c^3 + \frac{B - B'}{2}c^2 + (C - C')\,c + (D - D') = 0 \quad \ldots \ldots \quad (5),$$

$$\frac{A - A'}{2}c^2 + (B - B')\,c + (C - C') = 0 \quad \ldots \ldots \ldots \quad (6).$$

Again, when $x = c$,

$$d\mathrm{L}/dx - d\mathrm{R}/dx = -\,\omega^2 y . \mathrm{W}/g \quad (\S\,7,\ \text{equation }(5)),$$

and

$$\mathrm{L} - \mathrm{R} = -\,\omega^2 \mathrm{I}'\,dy/dx \quad (\S\,7,\ \text{equation }(6))\,;$$

whence

$$A - A' = -\frac{W\omega^2}{g\mathrm{EI}}\left\{\frac{A}{6}c^3 + \frac{B}{2}c^2 + Cc + D\right\} \quad \ldots \ldots \quad (7),$$

and

$$(A - A')\,c + (B - B') = -\frac{\omega^2 \mathrm{I}'}{\mathrm{EI}}\left(\frac{A}{2}c^2 + Bc + C\right) \quad \ldots \quad (8).$$

The elimination of the seven ratios

$$A : B : C : D : A' : B' : C' : D'$$

from the equations marked leads to the equation

$$\alpha^2 k^2 + 3\left\{\frac{\alpha l}{cc'} - \alpha k^2\left(\frac{1}{c^3} + \frac{1}{c'^3}\right)\right\} - \frac{9l^2}{c^3 c'^3} = 0 \quad \ldots \ldots \quad [A],$$

in which

$$\alpha = \mathrm{W}\omega^2/g\mathrm{EI}, \quad k = \sqrt{(g\mathrm{I}'/\mathrm{W})} \quad (\text{see } \S\,23,\ \text{p. } 305).$$

Hence,

$$k^2 = \frac{3l}{\alpha cc'} \cdot \frac{1 - \alpha\,(c^2c'^2/3l)}{\alpha\,\dfrac{c^2c'^2}{3l} - \left(\dfrac{c}{c'} + \dfrac{c'}{c} - 1\right)},$$

so that, for whirling to be at all possible (see Case IX., § 24, p. 304), $\alpha c^2c'^2/3l$ must be > 1 and $< c/c' + c'/c - 1$.

If $\alpha c^2c'^2/3l$ be equal to the first or second of these quantities, the corresponding value of ω is the inferior or superior limit of the speed respectively. Moreover, *the period of whirl corresponding to the inferior limit of speed is identical with the natural period of vibration of the light shaft under the given conditions.**

The superior limit is the inferior limit multiplied by some function of the position of the pulley. With the same pulley on the same shaft the superior limit = inferior limit $\times \sqrt{(c/c' + c'/c - 1)}$.

* This may be seen as follows :—

If W be the weight of the pulley, and ϵ the force necessary to deflect it one foot, then t (the time of lateral vibration) is $2\pi\sqrt{(W/g\epsilon)}$. To get ϵ, if P be the load acting at distances c, c' from the bearings, as in fig. 13, M the bending moment at a distance x from the shoulder, then (fig. 12) $M = x.Wc'/l$ from A to B, and $(l - x).Wc/l$ from B to C. Hence

$$d^2y/dx^2 = x.Pc'/lEI, \text{ from A to B},$$
$$d^2y'/dx^2 = (l - x).Pc/lEI, \text{ from B to C},$$

E and I having the same meanings as in the text.

We get, therefore,

$$y = \frac{Pc'}{lEI} \cdot \frac{x^3}{6} + Ex + F, \text{ from A to B} \quad \ldots \quad (1),$$

$$y' = \frac{Pc}{lEI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + E'x + F', \text{ from B to C} \quad \ldots \quad (2).$$

When

$$x = 0, \quad y = 0\,; \quad \text{therefore} \quad F = 0 \quad \ldots \quad (3).$$

When

$$x = l, \quad y' = 0\,;$$

therefore

$$F' = -\frac{Pcl^2}{3EI} - E'l \quad \ldots \quad (4).$$

When

$$x = c, \quad y = y', \quad \text{and} \quad dy/dx = dy'/dx\,;$$

therefore

$$(F - F') + (E - E')c - \frac{P}{EI} \cdot \frac{c^3}{3} = 0 \quad \ldots \quad (5),$$

$$(E - E') - \frac{P}{EI} \cdot \frac{c^2}{2} = 0 \quad \ldots \quad (6).$$

From equations (3), (4), (5), and (6), we get

If $c = c'$, that is to say, if the pulley be placed in the middle of the span, the superior and inferior limits coincide, and the pulley, at all speeds, revolves in a plane perpendicular to the original alignment of the shaft. Whatever be the size of the pulley, the period of whirling is the same as the "natural period of vibration."

The solution to equation [A] may be put in the form

$$\omega^2 = \frac{3g\text{EI}}{2\text{W}c^3}\left[\left\{\left(1 + \frac{b}{1-b}\Big|^3\right) - \frac{a^2}{1-b}\right\} + \sqrt{\left\{\left(1 + \frac{b}{1-b}\Big|^3\right) - \frac{a^2}{1-b}\right\}^2 + \frac{4a^2b}{(1-b)^3}}\,\right] \quad \text{[B]},$$

in which

$a = c/k =$ ratio of the distance of the pulley from the nearer bearing to the radius of gyration, and

$b = c/l =$ ratio of the distance of the pulley from the nearer bearing to the whole span.

Assuming certain values for a and b, results might be obtained giving relations between ω, W, c. In the equation [B], the ratio b fixes the position of the pulley on

$$\text{F}' = \frac{\text{P}c^3}{6\text{EI}} \quad \ldots \ldots \quad (7).$$

$$\text{E}' = -\frac{\text{P}c}{3\text{EI}}\left(l + \frac{c^2}{2l}\right) \quad \ldots \ldots \quad (8),$$

whence, substituting in (2) we get

$$y' = \frac{\text{P}c}{l\text{EI}}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) - \frac{\text{P}cx}{3\text{EI}}\left(l + \frac{c^2}{2l}\right) + \frac{\text{P}c^3}{6\text{EI}}.$$

Hence, when

$$x = c,$$

deflection

$$= \frac{\text{P}c^2c'^2}{3l\text{EI}}.$$

This is unity when $\text{P} = \epsilon$, wherefore

$$\epsilon = \frac{3l\text{EI}}{c^2c'^2}.$$

Hence

$$t = 2\pi\sqrt{\frac{\text{W}}{g\epsilon}} = 2\pi\sqrt{\frac{\text{W}c^2c'^2}{3gl\text{EI}}}.$$

The corresponding value of ω in text is

$$\omega = \frac{2\pi}{t} = \sqrt{\frac{3gl\text{EI}}{\text{W}c^2c'^2}},$$

therefore

$$\left(\frac{\text{W}\omega^2}{g\text{EI}}\right)\left(\frac{c^2c'^2}{3l}\right) = 1,$$

or

$$\alpha c^2c'^2/3l = 1,$$

as in text.

the shaft; and that being determined upon, the ratio a will fix the size of the pulley. For the same value of b, therefore, we should have different values of a.

The following are the results obtained, in this manner, from equation [B]. The vertical columns give the values of θ for different values of a, the value of b being fixed; whilst the rows denote the values of θ for different values of b, the value of a being kept the same.

27. *Values of θ in the equation $\omega = \theta\sqrt{(g\text{EI}/\text{W}c^3)}$, c being the distance of the pulley from the nearer bearing.*

		Values of $b = c/l$.							
		Very small.	$\frac{1}{10}$.	$\frac{1}{8}$.	$\frac{1}{6}$.	$\frac{1}{5}$.	$\frac{1}{4}$.	$\frac{1}{3}$.	$\frac{1}{2}$.
Values of $a = c/k$.	Superior limit	1·732	1·734	1·736	1·738	1·747	1·764	1·837	2·450
	·25	1·677	1·678	1·680	1·683	1·691	1·724	1·813	2·450
	·50	1·500	1·516	1·523	1·540	1·570	1·619	1·753	2·450
	·75	1·145	1·267	1·282	1·336	1·396	1·488	1·686	2·450
	1·00	0	·978	1·048	1·153	1·247	1·381	1·633	2·450
	1·25	do.	·819	·908	1·040	1·151	1·310	1·596	2·450
	1·50	do.	·787	·835	·970	1·095	1·266	1·572	2·450
	1·75	do.	·700	·795	·940	1·055	1·237	1·555	2·450
	2·00	do.	·676	·770	·916	1·038	1·212	1·543	2·450
	inferior limit	do.	·609	·699	·848	·969	1·155	1·500	2·450

It may be pointed out that when l is very large, and the pulley near the bearing, so that c/l is very small, the inferior limit for the case of the overhanging shaft (Case IX., § 25) is the superior limit for the case of pulley on a shaft resting on two bearings, the value of c being the same in both cases. The superior limit varies from 2·85 times the inferior limit to equality with it; and as the pulley is removed from the bearing to the centre of the span, the limits between which whirling is possible approximate more closely to each other.

28. *Experimental Results.*

The results, as given by equation [A.], page 308, merely take account of the effect of one pulley, the effect of the shaft and of all other pulleys which it carries being neglected. *If* N_1, N_2 *be the separate speeds of whirl of the shaft and pulley, on the assumption that the effect of one is neglected when that of the other is under consideration, then it is shown in* § 62, *page* 357, *that the resulting speed of whirl due to both causes combined may be taken to be of the form*

$$N_1N_2/\sqrt{(N_1^2 + N_2^2)}.$$

If the resulting speed given by this formula does not sufficiently approximate to the observed speed, then, by the introduction, in the terms of the denominator, of constant multipliers (which are determined by experiment), it will be shown, as occasion arises, that the speed given by the formula may be made to sufficiently approximate to the actual speed in all cases. Generally, however, the resulting speed given by the formula

$$N_1N_2/\sqrt{(N_1^2 + N_2^2)}$$

will be found to give results sufficiently accurate for practical purposes.

29. The following are the mean results of experiments made with pulleys I. and II. (see Chapter I., § 6, page 285) in different positions. The shaft, without the pulley, has been investigated in §§ 10, 11; whilst the calculated speeds for the pulleys alone have been obtained from equation [A.], page 308. The resulting calculated speed is taken to be $N_1N_2/\sqrt{(N_1^2 + N_2^2)}$ where N_1, N_2 are the separate speeds of whirl of the shaft and pulley. In all cases the distances c, c' are measured from the centre of the bearings to the centre of the pulley.

Pulley I.

Number of Experiment.	Date.	Conditions.			*Observed speed.*	Calculated speed, neglecting pulley and merely taking account of shaft (= N_1).	Calculated speed, neglecting shaft and merely taking account of pulley (= N_2).	*Resulting calculated speed.*	Percentage error.
		l in inches.	c in inches.	c' in inches.					
	1892.								
48	Oct. 26	32·00	1·00	31·00	1150	1121	13537	1117	+ 2·9
46	,,	32·00	2·91	29·10	1123	1121	4621	1089	+ 3·0
47	,,	32·00	4·00	28·00	1101	1121	3563	1069	+ 2·9
45	,,	32·00	5·33	26·66	1044	1121	2705	1036	+ ·8
44	,,	32·00	10·66	21·33	952	1121	1683	933	+ 2·0
43	Oct. 24	32·00	16·00	16·00	921	1121	1495	897	+ 2·6

PULLEY II.

Number of Experiment.	Date.	Conditions. l in inches.	c in inches.	c' in inches.	*Observed speed.*	Calculated speed, neglecting pulley and merely taking account of shaft (= N_1).	Calculated speed, neglecting shaft and merely taking account of pulley (= N_2).	*Resulting calculated speed.*	Percentage error.
	1892.								
54	Nov. 4	32·00	1·00	31·00	1130	1121	10355	1115	+ 1·3
53	"	32·00	2·91	29·10	1046	1121	3116	1055	− ·8
52	"	32·00	4·00	28·00	1007	1121	2389	1013	− ·6
51	Nov. 3	32·00	5·33	26 66	942	1121	1808	953	− 1·1
50	"	32·00	10·66	21·33	803	1121	1122	793	+ 1·2
49	Nov. 2	32·00	16·00	16·00	769	1121	997	745	+ 3·1

It will be noticed that in no case does the error materially exceed 3 per cent. of the observed speed. From Experiments 48, 46, and 54 it would appear that with the pulley near the bearing the pulley stiffens the shaft. That is to say, the shaft would whirl at a lower speed without the pulley than with it. The resulting calculated speed given above (viz., $N_1N_2/\sqrt{(N_1^2 + N_2^2)}$) must, of necessity, be less than either N_1 or N_2. If, however, the resulting calculated speed be taken to be $N_1N_2/\sqrt{(N_1^2 + aN_2^2)}$, where a is some constant determined from the experiments, then when N_2 is large compared to N_1, the resulting speed is $N_1/\sqrt{a}$, and if a be less than unity this would be greater than N_1. In this way all the calculated results could be made higher than those given above, so that in the experiments on Pulley I. (since the observed is greater than the calculated speed throughout) the observed and calculated results could be made to differ very slightly from one another. As, however, the errors in the experiments on Pulley II. are sometimes positive and sometimes negative, the resulting speed given by $N_1N_2/\sqrt{(N_1^2 + N_2^2)}$ is sufficiently near the actual speed for practical purposes.

30. The following are the mean results of experiments with both pulleys (I. and II.) on the shaft at the same time. *It is shown in Case XVII., §§ 60, 61, that the only way to deal with two or more pulleys is to consider each separately and then obtain the resulting speed of whirl by a formula similar to that in §§ 28 or 62.* The case of the shaft only is considered in §§ 10, 11; whilst the calculated results for each of the pulleys (considered separately) are obtained from the preceding article. *The resulting calculated speed is taken to be*

$$N_1N_2N_3/\sqrt{(N_1^2N_2^2 + N_2^2N_3^2 + N_3^2N_1^2)},$$

where N_1, N_2, N_3 *are the speeds of whirl for the shaft, Pulley I. and Pulley II., taken separately.* The span in all the experiments was 2′ 8″.

PULLEYS I. and II.

Number of experiment.	Date.	*Conditions.* Pulley I. c_1 in inches.	c_1' in inches.	Pulley II. c_2 in inches.	c_2' in inches.	*Observed speed.*	Calculated speed for shaft only.	Calculated speed for Pulley I. only.	Calculated speed for Pulley II. only.	*Resulting calculated speed.*	Percentage error.
	1892.										
69	Nov. 12	1·00	31·00	31·00	1·00	1118	1121	13537	10355	1111	+ ·6
70	,,	2·91	29·10	31·00	1·00	1099	1121	4621	10355	1083	+1·4
71	,,	4·00	28·00	31·00	1·00	1072	1121	3563	10355	1063	+ ·8
72	,,	5·33	26·66	31·00	1·00	1033	1121	2705	10355	1030	+ ·3
73	Nov. 14	10·66	21·33	31·00	1·00	955	1121	1683	10355	929	+2·7
74	,,	16·00	16·00	31·00	1·00	896	1121	1495	10355	893	+ ·3
75	,,	21·33	10·66	31·00	1·00	947	1121	1683	10355	929	+1·9
76	,,	26·66	5·33	31·00	1·00	1047	1121	2705	10355	1030	+1·6
77	Nov. 15	28·00	4·00	31·00	1·00	1067	1121	3563	10355	1064	+ ·3
78	Nov. 16	28·00	4·00	29·10	2·91	1033	1121	3563	3116	1012	+2·0
79	,,	21·33	10·66	29·10	2·91	920	1121	1683	3116	894	+2·8
80	,,	16·00	16·00	29·10	2·91	885	1121	1495	3116	862	+2·6
81	Nov. 17	10·66	21·33	29·10	2·91	925	1121	1683	3116	894	+3·3
82	,,	5·33	26·66	29·10	2·91	1030	1121	2705	3116	983	+4·6

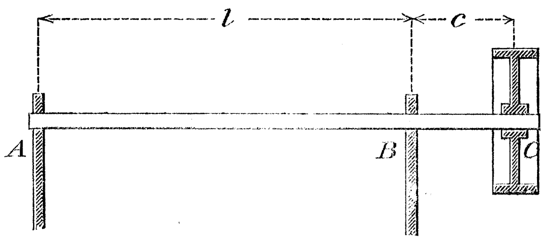
These experiments show that the method adopted in calculating the final resulting speed gives results very little different from the observed results. In all cases the percentage error is positive, so that the resulting speed, as calculated above, is slightly below the actual speed, and, consequently, errs on the right side.

Case XI.

31. SHAFT RESTING ON TWO SUPPORTS, l FEET APART, AND OVERHANGING ON ONE SIDE c FEET, LOADED WITH A PULLEY, WEIGHT W AND MOMENT OF INERTIA I′ AT ITS END.

Thus—

Fig. 14.



Take the origin at B. Then from A to B we have (§ 21, p. 304, equation 2)

$$y = \frac{A}{6}x^3 + \frac{B}{2}x^2 + Cx + D,$$

and from B to C

$$y' = \frac{A'}{6}x^3 + \frac{B'}{2}x^2 + C'x + D'.$$

When

$$x = 0, \qquad y = 0, \qquad y' = 0,$$
$$dy/dx = dy'/dx, \qquad d^2y/dx^2 = d^2y'/dx^2\,;$$

whence

$$D = 0 \quad . \quad . \quad . \quad . \quad (1),$$

$$D' = 0 \quad . \quad . \quad . \quad . \quad (2),$$

$$C = C' \quad . \quad . \quad . \quad . \quad (3),$$

$$B = B' \quad . \quad . \quad . \quad . \quad (4).$$

At A, where $x = -l$,

$$y = 0, \qquad d^2y/dx^2 = 0,$$

therefore

$$-\frac{A}{6}l^3 + \frac{B}{2}l^2 - Cl + D = 0 \quad . \quad . \quad . \quad . \quad (5),$$

$$-Al + B = 0 \quad . \quad . \quad . \quad . \quad (6).$$

When $x = c$ (at the pulley)

$$dL/dx - dR/dx = -\omega^2 y\,.\,W/g \quad (\S\ 7,\ \text{equation}\ (5)),$$

and

$$L - R = -\omega^2 I'\,dy/dx \quad (\S\ 7,\ \text{equation}\ (6))\,;$$

whence

$$A' = -\frac{W}{gEI}\omega^2\left(\frac{A'}{6}c^3 + \frac{B'}{2}c^2 + C'c + D'\right) \quad . \quad . \quad . \quad . \quad (7),$$

and

$$A'c + B' = -\frac{\omega^2 I'}{EI}\left(\frac{A'}{2}c^2 + B'c + C'\right) \quad . \quad . \quad . \quad . \quad (8).$$

The elimination of the seven ratios—

$$A : B : C : D : A' : B' : C' : D'$$

from these eight equations leads to

$$\tfrac{1}{12}\alpha^2\,.\,k^2c^3\left(c + \tfrac{4}{3}l\right) + \alpha\left\{\tfrac{1}{3}c^2(c + l) - k^2\left(c + \tfrac{1}{3}l\right)\right\} - 1 = 0 \quad . \quad . \quad [A],$$

in which

$$\alpha = W\omega^2/gEI \quad \text{and } k = \sqrt{(gI'/W)} \quad (\text{see } \S\ 23,\ \text{p. } 305).$$

If, in equation [A], we put $l = 0$, we get

$$\tfrac{1}{12}\,\alpha^2 k^2 c^4 + \alpha\left(\tfrac{1}{3}\,c^3 - ck^2\right) - 1 = 0,$$

the equation already obtained for the case of an overhanging shaft fixed in direction at one end. (Case IX., p. 304.)

From equation [A] we get

$$k^2 = \frac{1}{\alpha c}\cdot\frac{c - \frac{1}{3}\,\alpha c^3\,(c + l)}{\frac{1}{12}\,\alpha c^3\,(c + \frac{4}{3}\,l) - (c + \frac{1}{3}\,l)},$$

so that (as in § 24, p. 305) for whirling to be at all possible

$$\alpha c^3 \text{ must be} > 3c/c + l \text{ and } < 12\,(3c + l)/(3c + 4l).$$

If αc^3 be equal to the first or second of these quantities the corresponding value of ω gives the inferior or superior limit of the speed respectively. Moreover, *the period of whirl corresponding to the inferior limit of speed is identical with the natural period of vibration of the light shaft under the given conditions.*

The superior limit is the inferior limit multiplied by some function of the position of the pulley, that is, some function of l and c. The

$$\text{superior limit} = 2 \times \text{inferior limit} \times \sqrt{\left(\frac{3c + l}{3c + 4l}\cdot\frac{l + c}{c}\right)}.$$

If, as in Case X., p. 308, we put

$$a = c/k = \text{ratio of overhanging portion to the radius of gyration,}$$

and

$$b = c/l = \text{ratio of overhanging portion to the span,}$$

then the solution to the equation [A] is

$$\alpha c^3 = \frac{6}{3b + 4}\left[(3b+1) - a^2\,(b+1) + \sqrt{\{(3b+1) - a^2\,(b+1)\}^2 + a^2 b\,(3b+4)}\right] \quad . \quad [\text{B}].$$

32. The following are the results obtained from this equation by assuming certain values for a and b, as in Case X., § 27, p. 309. The vertical columns give the value of θ for different values of a, the value of b being fixed, whilst the rows denote the value of θ for different values of b, the value of a being kept the same.

Values of θ in the Equation $\omega = \theta\sqrt{(g\text{EI}/\text{W}c^3)}$, *c being the length of the overhanging portion.*

Values of $a = c/k$.		Values of $b = c/l$.						
		Very small.	$\frac{1}{10}$.	$\frac{1}{8}$.	$\frac{1}{6}$.	$\frac{1}{5}$.	$\frac{1}{4}$.	Very large.
	Superior limit	1·732	1·905	1·942	2·000	2·043	2·103	3·464
	·25	1·677	1·857	1·897	1·956	2·000	2·062	3·422
	·50	1·500	1·712	1·756	1·822	1·871	1·938	3·356
	·75	1·145	1·456	1·512	1·595	1·653	1·732	3·225
	1·00	0	1·111	1·188	1·297	1·370	1·464	3·048
	1·25	do.	·837	·920	1·037	1·116	1·217	2·841
	1·50	do.	·706	·782	·889	·963	1·058	2·628
	1·75	do.	·644	·713	·812	·880	·968	2·437
	2·00	do.	·609	·674	·769	·832	·923	2·282
	Inferior limit	do.	·522	·577	·655	·707	·774	1·732

Comparing these results with those in Case X., § 27, it will be noticed that when the span is very long and the pulley near the bearing, so that c/l is very small, the whirling speeds in the two cases are the same for the same values of c/k, whether the pulley be placed between the bearings, or overhang an equal distance on one side. For any other value of c/l, the superior limit in the present case is greater, and the inferior limit less, than the corresponding limit in Case X., the values of c and l being the same in both cases.

In the present case the superior limit varies from 3·65 times the inferior limit to twice that limit (when $c/l = \infty$, *i.e.*, the shaft works in a shoulder at one end).

33. *Experimental Results.*—The same remarks apply here as in § 28, page 310.

The following are the results of experiments made (1) with Pulley I. (p. 285), and (2) with Pulley II. at the end of the overhanging portion, the ratio of the overhanging portion of the span being made to vary. The shaft without the pulley has been investigated in §§ 12, 13; whilst the calculated speeds for the pulleys alone have been calculated from equation A, § 31, page 313. The calculated speed obtained from the formula $N_1N_2/\sqrt{(N_1^2 + N_2^2)}$ where N_1, N_2 are the separate speeds of whirl for the shaft and pulley, gives results, in every case, much lower than the observed results. *To bring the calculated results more in accordance with the observed results, the resulting calculated speed is taken to be* $N_1N_2/\sqrt{(N_1^3 + N_2^2 a)}$, *the value of a determined from Experiment* 64—*chosen because the observed and calculated values of the whirling speed for the shaft alone are practically the same* (see § 13, page 294, Experiment 24)—*being* ·885. In this expression, a is the multiplier of the greatest term in the denominator.

PULLEY I.

Number of experiment.	Date.	Conditions. l in inches.	Conditions. c in inches.	*Observed speed.*	Calculated speed for shaft only (N_1).	Calculated speed for pulley only (N_2).	Calculated speed by formula $N_1N_2/\sqrt{(N_2^2 + N_2^2)}$.	Corresponding percentage error.	*Resulting calculated speed* by formula $N_1N_2/\sqrt{(N_1^2 + aN_2^2)}$.	*Percentage error.*
	1892.									
63	Nov. 11	30·70	1·00	1223	1175	16390	1170	+ 4·3	1246	− 1·9
64	,,	29·10	2·61	1329	1301	4808	1256	+ 5·5	1329	0·0
65	,,	28·00	3·69	1384	1397	3318	1288	+ 6·9	1410	− 1·9
66	,,	26·66	5·02	1407	1516	2428	1286	+ 8·6	1343	+ 4·5
67	,,	24·00	7·69	1224	1704	1572	1156	+ 5·5	1199	+ 2·0
68	,,	21·33	10·35	968	1606	1162	941	+ 2·8	979	− 1·1

PULLEY II.

Number of experiment.	Date.	Conditions. l in inches.	Conditions. c in inches.	*Observed speed.*	Calculated speed for shaft only (N_1).	Calculated speed for pulley only (N_2).	Calculated speed by formula $N_1N_2/\sqrt{(N_1^2 + N_2^2)}$.	Corresponding percentage error.	*Resulting calculated speed* by formula $N_1N_2/\sqrt{(N_1^2 + aN_2^2)}$.	*Percentage error.*
	1892.									
62	Nov. 9	30·63	1·00	1227	1175	13816	1173	+ 4·6	1224	+ 1·4
57	,, 7	29·10	2·54	1276	1301	3353	1213	+ 5·0	1278	0·0
58	,, 7	28·00	3·63	1281	1397	2277	1191	+ 7·0	1256	+ 1·9
59	,, 7	26·66	4·96	1215	1516	1643	1114	+ 8·3	1150	+ 5·3
60	,, 9	24·00	7·63	928	1704	1056	898	+ 3·2	937	− 1·0
61	,, 9	21·33	10·29	712	1606	782	703	+ 1·1	738	− 3·6

These experiments show that the same value of a (= ·885) in the formula for the resulting speed—viz., $N_1N_2/\sqrt{(N_1^2 + aN_2^2)}$ when $N_2 > N_1$, and $N_1N_2/\sqrt{(N_2^2+aN_1^2)}$ when $N_1 > N_2$—holds, to a sufficient degree of approximation, whatever be the ratio of the overhanging portion to the span, or whatever be the size of the pulley.

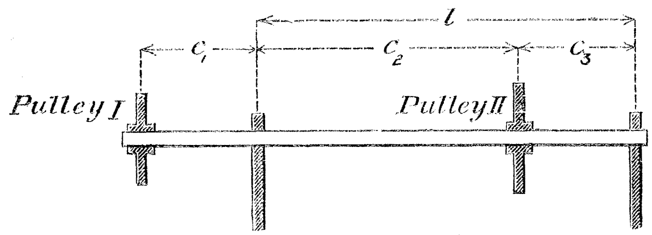
Moreover, as in § 29, p. 311, when the pulley is near the bearing the shaft is stiffened by the pulley, and the lighter the pulley the further the distance which it might be from the bearing before this stiffening action ceases. (Compare § 29, Experiments 48, 46, 54, and present article Experiments 63, 64, 62.)

34. The following are the mean results of experiments with both pulleys, I. and II., on the shaft. It is shown in Case XVII., §§ 59–62, pp. 76–80, that the only way to deal with two or more pulleys is to consider each separately and obtain the resulting whirling speed from a formula of the type

$$N_1N_2N_3/\sqrt{(N_1^2N_2^2 + N_2^2N_3^2 + N_3^2N_1^2)},$$

where N_1, N_2, N_3 are the separate speeds of whirl due to the several causes on the assumption that each cause is neglected except the one under consideration. In the present series of experiments Pulley I. was kept at the end of the overhanging portion, whilst Pulley II. was placed in different positions between the bearings, the distance between the bearings being also altered. The notation used in the following results will be understood from the diagram.

Fig. 15.



To get the resulting calculated speed, the resulting speed for the shaft (N_1), and for Pulley I. (N_2) is obtained, as explained in the preceding article. Let this be called N_3. The whirling speed for Pulley II. alone is obtained from equation A., § 26, p. 308. Let this be N_4. Then the resulting speed for both the pulleys and the shaft combined is taken to be

$$\frac{N_3N_4}{\sqrt{(N_3^2 + N_4^2)}} \quad \text{or} \quad \frac{N_1N_2N_4}{\sqrt{(N_1^2N_2^2 + aN_2^2N_4^2 + N_4^2N_1^2)}}.$$

PULLEYS I. and II.

Number of experiment.	Date.	Conditions. c_1 in inches.	l in inches.	c_2 in inches.	c_3 in inches.	*Observed speed.*	Calculated speed for shaft only (N_1).	Calculated speed for Pulley I. only (N_2).	Calculated speed for Pulley II. only (N_4).	Calculated speed for shaft and Pulley I. ($N_3 = N_1^2N_2^2/\sqrt{N_1^2 + aN_2^2}$).	*Resulting calculated speed* ($= N_3N_4/\sqrt{N_3^2 + N_4^2}$).	Percentage error.
83	1892 Nov. 25	1·00	30·69	1·00	29·69	1225	1171	16390	13360	1246	1241	−1·3
84	,, ,,	1·00	30·69	4·00	26·69	1117	1171	16390	2377	1246	1104	+1·2
85	,, ,,	1·00	30·69	9·00	21·69	897	1171	16390	1277	1246	892	+ ·5
86	,, 28	1·00	30·69	15·35	15·35	811	1171	16390	1061	1246	808	+ ·4
87	,, ,,	1·00	30·69	21·69	9·00	897	1171	16390	1277	1246	892	+ ·5
88	,, ,,	1·00	30·69	26·69	4·00	1102	1171	16390	2377	1246	1104	·0
93	Nov. 30	2·60	29·10	1·00	28·10	1318	1301	4808	13676	1329	1323	− ·4
92	,, ,,	2·60	29·10	4·00	25·10	1189	1301	4808	2460	1329	1169	+1·7
91	,, ,,	2·60	29·10	9·00	20·10	965	1301	4808	1346	1329	946	+1·9
90	,, 28	2·60	29·10	14·50	14·5	884	1301	4808	1149	1329	869	+1·7
89	,, ,,	2·60	29·10	25·10	4.00	1180	1301	4808	2460	1329	1169	+ ·9
94	Nov. 30	5·01	26·66	1·00	25·66	1362	1516	2428	14202	1343	1337	+1·8
95	Dec. 1	5·01	26·66	4·00	22·66	1251	1516	2428	2607	1343	1194	+4·6
96	,, ,,	5·01	26·66	8·00	18·66	1055	1516	2428	1563	1343	1019	+3·4
97	,, ,,	5·01	26·66	13·33	13·33	955	1516	2428	1310	1353	938	+1·8

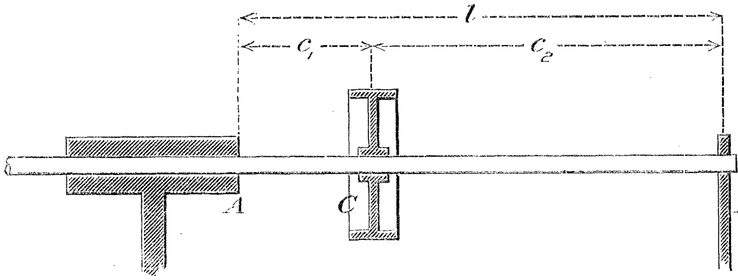
These experiments show that the method adopted in calculating the final resulting speed gives results approximating very closely to the observed speeds. When the overhanging portion is small, and both pulleys are near the bearings, the shaft is stiffened by the pulleys (Experiment 83, 93). In all other cases the resulting speed as calculated above is slightly below the actual speed. The formula consequently errs on the right side.

Case XII.

35. SHAFT, LENGTH l, RESTING FREELY ON A SUPPORT AT ONE END AND FIXED IN DIRECTION AT THE OTHER, LOADED WITH A PULLEY, WEIGHT W AND MOMENT OF INERTIA I′, PLACED AT A DISTANCE c_1 FROM THE SHOULDER END, AND c_2 FROM THE FREE END.

Thus—

Fig. 16.



We have (§ 21, equation 2), taking the origin at the shoulder end A,

$$y = \frac{A}{6} x^3 + \frac{B}{2} x^2 + Cx + D$$

from A to the pulley, and

$$y' = \frac{A'}{6} x^3 + \frac{B'}{2} x^2 + C'x + D'$$

from the pulley to B.

When $x = 0$,

$$y = 0, \qquad dy/dx = 0\,;$$

whence

$$D = 0 \quad \ldots \ldots \ldots \quad (1),$$

$$C = 0 \quad \ldots \ldots \ldots \quad (2).$$

When $x = c$,

$$y = y', \qquad dy/dx = dy'/dx\,;$$

whence

$$\tfrac{1}{6}(A - A')\,c_1^{\,3} + \tfrac{1}{2}(B - B')\,c_1^{\,2} + (C - C')\,c_1 + (D - D') = 0 \quad \ldots \quad (3),$$

$$\tfrac{1}{2}(A - A')\,c_1^{\,2} + (B - B')\,c_1 + (C - C') = 0 \quad \ldots \ldots \quad (4).$$

When $x = l$,

$$y' = 0, \qquad d^2y'/dx^2 = 0,$$

whence

$$\tfrac{1}{6}\,A'\,l^3 + \tfrac{1}{2}\,B'\,l^2 + C'l + D' = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (5),$$

$$A'l + B' = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6).$$

When $x = c_1$ (at C), we have

$$dL/dx - dR/dx = -\,W/g\,.\,\omega^2 y \quad (\S\,7,\text{ equation } (5)),$$

and

$$L - R = -\,\omega^2 I'\,dy/dx \quad (\S\,7,\text{ equation } (6))\,;$$

whence we obtain, putting as before (§ 23, p. 305)

$$\alpha = W\omega^2/gEI, \quad \beta = \omega^2 I'/EI, \quad \text{and} \quad \beta = \alpha k^2, \text{ where } k^2 = \sqrt{(gI'/W)}.$$

$$(A - A') = -\,\alpha\,(\tfrac{1}{6}\,A\,c_1^3 + \tfrac{1}{2}\,B\,c_1^2 + Cc_1 + D) \quad . \; . \; . \; . \; . \quad (7),$$

$$(A - A')\,c_1 + (B - B') = -\,\beta\,(\tfrac{1}{2}\,A\,c_1^2 + Bc_1 + C) \quad . \; . \; . \quad (8).$$

The elimination of the seven ratios

$$A : B : C : D : A' : B' : C' : D'$$

from the eight equations marked leads to

$$\alpha^2\,\frac{k^2 c_1^4 c_2^3}{36} + \alpha\left\{\frac{c_1^3 c_2^2}{6}\left(\frac{c_1}{2} + \frac{2c_2}{3}\right) - \frac{k^2 c_1}{3}\left(c_2^3 + \frac{c_1^3}{4}\right)\right\} - \frac{l^3}{3} = 0 \quad . \; . \quad [A],$$

a quadratic in ω^2, which is not, of course, symmetrical with respect to c_1, c_2.

If $l = \infty$, then $c_2 = l$, and the equation reduces to

$$\alpha^2\,\tfrac{1}{12}\,k^2 c_1^4 + \alpha\,(\tfrac{1}{3}\,c_1^3 - k^2 c_1) - 1 = 0,$$

the equation already obtained for the case of an overhanging shaft working in a shoulder. (Case IX., § 23, p. 304.)

From equation [A] we get

$$k^2 = \frac{1}{\alpha c_1}\cdot\frac{l^3 - \alpha c_1^3 c_2^2\,(\tfrac{1}{4}c_1 + \tfrac{1}{3}c_2)}{\tfrac{1}{12}\,(\alpha c_1^3 c_2^3) - (c_2^3 + \tfrac{1}{4}c_1^3)}\,,$$

so that for whirling to be at all possible (see Case IX., § 24, p. 305) we must have

$$\alpha c_1^3 c_2^3 > l^3 c_2/(\tfrac{1}{4}c_1 + \tfrac{1}{3}c_2) \quad \text{and} \quad < 12\,(c_2^3 + \tfrac{1}{4}c_1^3).$$

If $\alpha c_1^3 c_2^3$ be equal to the first or second of these quantities, the corresponding value of ω gives the inferior or superior limit of the speed respectively. The values of k, corresponding to these limiting values of ω, are zero and infinity, and if the shaft whirl at the speeds given by them it will do so in such a manner that the pulley still rotates in a plane perpendicular to the original alignment of the shaft.

Moreover, *the period of whirl corresponding to the inferior limit of speed is identical with the natural period of vibration of the light shaft under the given conditions.*

The

$$\text{superior limit} = \text{inferior limit} \times \sqrt{\left\{\left(\frac{c_2^3}{l^3} + \frac{c_1^3}{4l^3}\right)\left(1 + 3\frac{l}{c_2}\right)\right\}}.$$

Let

$$a_1 = c_1/k, \qquad b_1 = c_1/l\,;$$

that is a_1 and b_1 are the ratios of the distance of the pulley from the shoulder end of the shaft to the radius of gyration of the pulley and to the span respectively. Also let a_2, b_2 be the corresponding ratios when the distance of the pulley is measured from the free end of the shaft; that is

$$a_2 = c_2/k \quad \text{and} \quad b_2 = c_2/l.$$

Then the solution to equation [A], p. 320, may be expressed in either of the forms

$$\tfrac{1}{6}\alpha c_1^3 = 1 + \tfrac{1}{4}\cdot\frac{b_1^3}{(1-b_1)^3} - a_1^2\left(\tfrac{1}{3} + \tfrac{1}{4}\cdot\frac{b_1}{1-b_1}\right)$$
$$+ \sqrt{\left[\left\{1 + \tfrac{1}{4}\cdot\frac{b_1^3}{(1-b_1)^3} - a_1^2\left(\tfrac{1}{3} + \tfrac{1}{4}\cdot\frac{b_1}{1-b_1}\right)\right\}^2 + \tfrac{1}{3}\cdot\frac{a_1^2}{(1-b_1)^3}\right]} \quad . \quad . \quad [\text{B}],$$

or

$$\tfrac{1}{6}\alpha c_2^3 = \left(\frac{b_2^3}{(1-b_2)^3} + \tfrac{1}{4}\right) - a_2^2\left(\tfrac{1}{4} + \tfrac{1}{3}\cdot\frac{b_2}{1-b_2}\right)$$
$$+ \sqrt{\left[\left\{\left(\frac{b_2^3}{(1-b_2)^3} + \tfrac{1}{4}\right) - a_2^2\left(\tfrac{1}{4} + \tfrac{1}{3}\cdot\frac{b_2}{1-b_2}\right)\right\}^2 + \tfrac{1}{3}\cdot\frac{a_2^2 b_2}{(1-b_2)^4}\right]} \quad . \quad . \quad (\text{C}).$$

As in Cases X. and XI. (§§ 27, 32), by assuming certain values for a_1, b_1, or a_2, b_2, the corresponding values of αc_1^3 or αc_2^3 can be found, and so, for any particular value of c_1 or c_2, the value of ω readily calculated. Two sets of results have thus been compiled. The first set (obtained from equation [B]) gives the values of αc_1^3 for different values of a_1 and b_1, and is applicable when the pulley lies between the shoulder end and the centre of the span; whilst the second set (obtained from equation [C]) gives values of αc_2^3 for different values of a_2 and b_2, and is applicable when the pulley lies between the free end and the centre of the span.

36. *Values of* θ_1 *in the equation* $\omega = \theta_1 \sqrt{(g\mathrm{EI}/\mathrm{W}c_1^3)}$, *when the pulley lies between the shoulder end and the centre of the span, and* c_1 = *distance of pulley from shoulder end.*

		Values of $b_1 = c_1/l$.					
		Very small	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
Values of $a_1 = c_1/k$.	Superior limit	3·464	3·465	3·467	3·480	3·518	3·873
	·25	3·437	3·438	3·441	3·457	3·498	3·868
	·50	3·356	3·359	3·366	3·388	3·396	3·855
	·75	3·225	3·233	3·247	3·284	3·361	3·838
	1·00	3·048	3·069	3·096	3·157	3·267	3·819
	1·25	2·841	2·885	2·933	3·026	3·173	3·800
	1·50	2·628	2·705	2·778	2·906	3·090	3·785
	1·75	2·437	2·549	2·646	2·805	3·021	3·772
	2·00	2·282	2·424	2·547	2·726	2·966	3·761
	Inferior limit	1·732	1·981	2·123	2·385	2·714	3·704

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37. *Values of* θ_2 *in the equation* $\omega = \theta_2 \sqrt{(g\text{EI}/\text{W}c_2^3)}$, *when the pulley lies between the free end and the centre of the span, and* c_2 = *distance of pulley from free end.*

		Values of $b_2 = c_2/l$.					
		Very small	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
Values of $a_2 = c_2/l$.	Superior limit	1·732	1·737	1·760	1·851	2·121	3·873
	·25	1·677	1·686	1·716	1·829	2·115	3·868
	·50	1·500	1·533	1·598	1·766	2·098	3·855
	·75	1·146	1·300	1·442	1·693	2·079	3·838
	1·00	0	1·075	1·309	1·633	2·063	3·819
	1·25	do.	·938	1·223	1·591	2·051	3·800
	1·50	do.	·866	1·167	1·562	2·043	3·785
	1·75	do.	·826	1·136	1·542	2·036	3·772
	2·00	do.	·801	1·114	1·529	2·030	3·761
	Inferior limit	do.	·728	1·050	1·475	2·012	3·704

When the span is very long and the pulley is near the shoulder, so that c_1/l may be taken to be very small, a comparison of the results in §§ 36 and 25 shows that the effect of the free end is *nil*; in other words, the speeds are the same as if the shaft merely overhung. If the pulley be near the free end of the span, so that c_2/l may be taken to be very small, a comparison of the results in §§ 37, 27, 32 shows that the effect of the shoulder is precisely the same as that of a free bearing. These results might, of course, have been anticipated.

38. Comparing these results with those obtained in Case X., § 27 (that is, with the case of a pulley on a shaft merely resting on a support at each end), we see that in the case where one end is fixed in direction, the calculated speed for the pulley

alone exceeds that in the case of a shaft free at both ends, in a certain ratio—that ratio depending on the position and size of the pulley.

Considering the superior limits in each case, the increase of speed due to the shoulder is 100 per cent. at the shoulder end, decreasing to 91 at one-third the span from the shoulder end, 58 at the centre of the span, and zero at the free end.

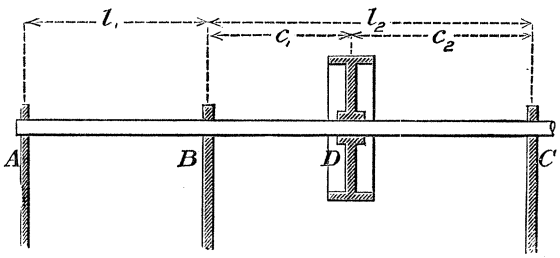
Considering the inferior limits in each case, the increase of speed is 225 per cent. near the shoulder end, 51 at the centre of the span, and 19 per cent. near the free end.

Case XIII.

39. Shaft supported on three bearings, l_1 and l_2 feet apart respectively and loaded with a pulley, weight W and moment of inertia I′ on the span of length l_2, the pulley being distant c_1 feet from the middle bearing and c_2 feet from the end bearing.

Thus :—

Fig. 17.



We have, taking the origin at the middle bearing B (§ 21, equation 2),

$$y = \frac{A}{6}x^3 + \frac{B}{2}x^2 + Cx + D, \quad \text{from A to B},$$

$$y' = \frac{A'}{6}x^3 + \frac{B'}{2}x^2 + C'x + D', \quad \text{from B to D},$$

and

$$y'' = \frac{A''}{6}x^3 + \frac{B''}{2}x^2 + C''x + D'', \quad \text{from D to C}.$$

When $x = 0$,

$$y = 0, \qquad y' = 0, \qquad dy/dx = dy'/dx, \qquad d^2y/dx^2 = d^2y'/dx^2\,;$$

whence we obtain

$$D = 0 \quad \ldots \ldots \quad (1),$$

$$D' = 0 \quad \ldots \ldots \quad (2),$$

$$C = C' \quad \ldots \ldots \quad (3),$$

$$B = B' \quad \ldots \ldots \quad (4).$$

When $x = -l_1$,

$$y = 0, \qquad d^2y/dx^2 = 0\,;$$

whence

$$-\tfrac{1}{6}\mathrm{A}l_1^3 + \tfrac{1}{2}\mathrm{B}l_1^2 - \mathrm{C}l_1 + \mathrm{D} = 0 \quad \ldots \ldots \quad (5),$$

$$-\mathrm{A}l_1 + \mathrm{B} = 0 \quad \ldots \ldots \quad (6).$$

When $x = l_2$,

$$y'' = 0, \qquad d^2y''/dx^2 = 0\,;$$

whence

$$\tfrac{1}{6}\mathrm{A}''l_2^3 + \tfrac{1}{2}\mathrm{B}''l_2^2 + \mathrm{C}''l_2 + \mathrm{D}'' = 0 \quad \ldots \ldots \quad (7),$$

$$\mathrm{A}''l_2 + \mathrm{B}'' = 0 \quad \ldots \ldots \quad (8).$$

At the pulley (D) we have, when $x = c_1$,

$$y' = y'', \qquad dy'/dx = dy''/dx,$$
$$d\mathrm{L}/dx - d\mathrm{R}/dx = -\,\omega^2 y'.\mathrm{W}/g \quad (\S\ 7,\ \text{equation } 5),$$
$$\mathrm{L} - \mathrm{R} = -\,\omega^2 \mathrm{I}'\,dy'/dx \quad (\S\ 7,\ \text{equation } 6)\,;$$

whence

$$\tfrac{1}{6}(\mathrm{A}' - \mathrm{A}'')\,c_1^3 + \tfrac{1}{2}(\mathrm{B}' - \mathrm{B}'')\,c_1^2 + (\mathrm{C}' - \mathrm{C}'')\,c_1 + (\mathrm{D}' - \mathrm{D}'') = 0 \quad \ldots \quad (9),$$

$$\tfrac{1}{2}(\mathrm{A}' - \mathrm{A}'')\,c_1^2 + (\mathrm{B}' - \mathrm{B}'')\,c_1 + (\mathrm{C}' - \mathrm{C}'') = 0 \quad \ldots \quad (10),$$

$$\mathrm{A}' - \mathrm{A}'' = -\,\alpha\,(\tfrac{1}{6}\mathrm{A}'c_1^3 + \tfrac{1}{2}\mathrm{B}'c_1^2 + \mathrm{C}'c_1 + \mathrm{D}') \quad \ldots \quad (11),$$

$$(\mathrm{A}' - \mathrm{A}'')\,c_1 + (\mathrm{B}' - \mathrm{B}'') = -\,\beta\,(\tfrac{1}{2}\mathrm{A}'c_1^2 + \mathrm{B}'c_1 + \mathrm{C}') \quad \ldots \quad (12),$$

where, as in § 23, p. 305,

$$\alpha = \mathrm{W}\omega^2/g\mathrm{EI}, \quad \beta = \omega^2\mathrm{I}'/\mathrm{EI} \quad \text{and } \beta = \alpha k^2, \text{ where } k = \sqrt{(g\mathrm{I}'/\mathrm{W})}.$$

The elimination of the eleven ratios A : B : C : D : A′ : B′ : C′ : D′ : A″ : B″ : C″ : D″ from the twelve equations marked leads to

$$\alpha^2\,\frac{k^2c_1^3c_2^3}{9}\left(\frac{c_1}{4} + \frac{l_1}{3}\right) + \alpha\left\{\frac{c_1^2c_2^2}{9}\left(l_1l_2 + c_1\,.\,\overline{c_2 + 4}\right) - k^2\left(\frac{c_1}{3}\,.\,\overline{c_2^3 + \frac{c_1^3}{4}} + \frac{l_1}{9}\,.\,\overline{c_1^3 + c_2^3}\right)\right\}$$
$$-\,\frac{l_2^2}{3}\,(l_1 + l_2) = 0 \quad \ldots \ldots \quad [\mathrm{A}],$$

a quadratic in ω^2 which is not symmetrical with respect to c_1, c_2.

If in equation [A] we put $l_1 = \infty$, it reduces to

$$\alpha^2\tfrac{1}{9}k^2c_1^3c_2^3 + \alpha\tfrac{1}{3}\{c_1^2c_2^2l_2 - k^2\,(c_1^3 + c_2^3)\} - l_2^2 = 0,$$

the equation already obtained for the case of a shaft resting freely on two supports at the ends, and loaded with a pulley distant c_1, c_2 from the bearings (Case X., § 26, p. 308).

If $l_1 = 0$, the equation reduces to

$$\alpha^2 \frac{k^2 c_1^4 c_2^3}{36} + \alpha \left\{ \frac{c_1^3 c_2^3}{9} \left(c_2 + \frac{3c_1}{4} \right) - k^2 \frac{c_1}{3} \left(c_2^3 + \frac{c_1^3}{4} \right) \right\} - \frac{l_2^2}{3} = 0,$$

the equation already obtained for the case of a shaft resting freely on a support at one end and working in a shoulder at the other (Case XII., § 35, p. 320).

If $l_2 = c_2 = \infty$ the equation reduces to

$$\alpha^2 \frac{k^2 c_1^3}{12} (c_1 + \tfrac{4}{3} l_1) + \alpha \left\{ \frac{c_1^2}{3} (l_1 + c_1) - k^2 \left(c_1 + \frac{l_1}{3} \right) \right\} - 1 = 0,$$

the equation already obtained for the case of a shaft, span l_1, and overhanging a distance c_1, the pulley being at the extremity (Case XI., § 31, p. 313).

40. In the case of two spans, one of which is loaded, it is, of course, useless to completely solve the many cases which might occur. The three cases which at once suggest themselves for full investigation are—

(1.) Unloaded span zero.

(2.) Unloaded span infinite.

(3.) Unloaded span equal to loaded span.

It has been shown that the first two cases have been already investigated (Cases XII. and X.). *It only remains to solve the third case when the two spans are equal.*

If

$$l_1 = l_2 = l,$$

equation [A] becomes

$$\alpha^2 \frac{k^2 c_1^3 c_2^3}{3} \left(\frac{c_1}{4} + \frac{l}{3} \right)$$

$$+ \alpha \left[\frac{c_1^2 c_2^2}{3} \left\{ l^2 + c_1 \left(c_2 + \frac{3c_1}{4} \right) \right\} - k^2 \left\{ c_1 \left(c_2^3 + \frac{c_1^3}{4} \right) + \frac{l}{3} (c_1^3 + c_2^3) \right\} \right] - 2l^3 = 0. \quad \text{[B]},$$

from which we immediately get

$$k^2 = \frac{1}{\alpha c_1 c_2} \cdot \frac{2\, c_1 c_2 l^3 - \alpha \frac{1}{3} c_1^3 c_2^3 (l^2 + c_1 . \overline{c_2 + \frac{3}{4} c_1})}{\frac{1}{3} \alpha c_1^3 c_2^3 (\frac{1}{4} c_1 + \frac{1}{3} l) - (c_1 . \overline{c_2^3 + \frac{1}{4} c_1^3} + \frac{1}{3} l . \overline{c_1^3 + c_2^3}},$$

so that for whirling to be at all possible we must have (see Case IX. § 24, p. 305),

$$\alpha \frac{c_1^3 c_2^3}{3} > \frac{2\, c_1 c_2 l^3}{l^2 + c_1 (c_2 + \frac{3}{4} c_1)}$$

and

$$< \frac{c_1(c_2^3 + \frac{1}{4}c_1^3) + \frac{1}{3}l(c_1^3 + c_2^3)}{\frac{1}{4}c_1 + \frac{1}{3}l}$$

If $\alpha c_1^3 c_2^3/3$ be equal to the first or second of these expressions, the corresponding value of ω gives the inferior or superior limit of the speed respectively. Moreover, *the period of whirl corresponding to the inferior limit of speed is identical with the natural period of vibration of the light shaft under the given conditions.*

The superior limit

$$= \text{inferior limit} \times \sqrt{\left(\frac{c_1(c_2^3 + \frac{1}{4}c_1^3) + \frac{1}{3}l(c_1^3 + c_2^3)}{\frac{1}{4}c_1 + \frac{1}{3}l} \times \frac{l^2 + c_1(c_2 + \frac{3}{4}c_1)}{2c_1c_2l^3}\right)}.$$

Let

$$a_1 = c_1/k \quad \text{and} \quad b_1 = c_1/l,$$

that is, a_1 and b_1 are the ratios of the distance of the pulley from the middle bearing to the radius of gyration of the pulley and to either span respectively. Also, let a_2, b_2 be the corresponding ratios when the distance of the pulley is measured from the end bearing; that is

$$a_2 = c_2/k \quad \text{and} \quad b_2 = c_2/l.$$

Then the solution to equation [B] may be put in either of the forms

$$\frac{(4 + 3b_1) \cdot \alpha c_1^3}{6} = 3b_1\left(1 + \frac{1}{4}\frac{b_1^3}{(1 - b_1)^3}\right) + 1 + \frac{b_1^3}{(1 - b_1)^3} - a_1^2\left\{\frac{1}{1 - b_1} + b_1\left(1 + \frac{3}{4}\frac{b_1}{1 - b_1}\right)\right\}$$
$$+ \sqrt{\left[3b_1\left(1 + \frac{1}{4}\frac{b_1^3}{(1 - b_1)^3}\right) + 1 + \frac{b_1^3}{(1 - b_1)^3} - a_1^2\left\{\frac{1}{1 - b_1} + b_1\left(1 + \frac{3}{4}\frac{b_1}{1 - b_1}\right)\right\}\right]^2 + a_1^2 \cdot \frac{2b_1(4 + 3b_1)}{(1 - b_1)^3}} \quad \text{[C]},$$

and

$$\frac{(7 - 3b_2) \cdot \alpha c_2^3}{6} = 3(1 - b_2)\left(\frac{1}{4} + \frac{b_2^3}{(1 - b_2)^3}\right) + 1 + \frac{b_2^3}{(1 - b_2)^3} - a_2^2\left\{\frac{1}{1 - b_2} + b_2 + \frac{3}{4}(1 - b_2)\right\}$$
$$+ \sqrt{\left[3(1 - b_2)\left(\frac{1}{4} + \frac{b_2^3}{(1 - b_2)^3}\right) + 1 + \frac{b_2^3}{(1 - b_2)^3} - a_2^2\left\{\frac{1}{1 - b_2} + b_2 + \frac{3}{4}(1 - b_2)\right\}\right]^2 + a_2^2 \cdot \frac{2b_2(7 - 3b_2)}{(1 - b_2)^3}} \quad \text{[D]}.$$

As in Cases X., XI., and XII. (§§ 27, 32, 36, 37), by assuming certain values for a_1, b_1, or a_2, b_2, the corresponding values of αc_1^3, or αc_2^3 can be found, and so, for any particular value of c_1 or c_2, the value of ω readily calculated. Two sets of results have thus been compiled. The first set (obtained from equation [C]) gives the values of αc_1^3 for different values of a_1 and b_1, and is applicable when the pulley lies between the middle bearing and the centre of the span; whilst the second set (obtained from equation [D]) gives values of αc_2^3 for different values of a_2 and b_2, and is applicable when the pulley lies between the end bearing and the centre of the span.

41. *Values of θ_1 in the equation $\omega = \theta_1 \sqrt{(g\text{EI}/\text{W}c_1^3)}$ when the pulley lies between the middle bearing and the centre of span, and c_1 = distance from mid-bearing.*

		Values of $b_1 = c_1/l$.					
		Very small	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
Values of $a_1 = c_1/k$.	Superior limit	1·732	1·906	2·006	2·129	2·275	2·908
	·25	1·677	1·860	1·966	2·098	2·254	2·907
	·50	1·500	1·725	1·853	2·012	2·199	2·905
	·75	1·146	1·511	1·687	1·897	2·131	2·902
	1·00	0	1·279	1·520	1·786	2·066	2·900
	1·25	do.	1·109	1·441	1·700	2·016	2·898
	1·50	do.	1·012	1·310	1·641	1·978	2·896
	1·75	do.	·956	1·257	1·599	1·952	2·895
	2·00	do.	·921	1·220	1·571	1·932	2·894
	Inferior limit	do.	·822	1·114	1·470	1·857	2·890

42. *Values of θ_2 in the equation $\omega = \theta_2 \sqrt{(g\mathrm{EI}/\mathrm{W}c_2^3)}$ when the pulley lies between the free end and the centre of the span, and c_2 = distance of pulley from free end.*

		Values of $b_2 = c_2/l$.					
		Very small	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
Values of $a_2 = c_2/k$.	Superior limit	1·732	1·735	1·747	1·795	1·936	2·908
	·25	1·677	1·682	1·700	1·764	1·920	2·907
	·50	1·500	1·524	1·567	1·676	1·880	2·905
	·75	1·146	1·273	1·385	1·578	1·837	2·902
	1·00	0	1·022	1·225	1·485	1·801	2·900
	1·25	do.	·872	1·123	1·428	1·775	2·898
	1·50	do.	·796	1·063	1·386	1·757	2·896
	1·75	do.	·755	1·027	1·365	1·745	2·895
	2·00	do.	·731	1·004	1·346	1·738	2·894
	Inferior limit	do.	·661	·932	1·285	1·671	2·890

By a comparison of these two sets of results it will be noticed that the same pulley placed at equal distances from the middle bearing and the end bearing of the shaft whirls at different speeds, those near the middle bearing being higher than those near the end bearing. Moreover, *if the span be very long and the pulley be near the bearing, so that c/l may be taken to be very small,* it will be seen that, whilst the superior limits in the two cases are the same, the ratio which the inferior limit bears to the superior limit is less when the pulley is near the end bearing than when it is near the middle bearing. Also the superior limits when the pulley is near either bearing are the same as those obtained in Case X., § 27, Case XI., § 32, and also in Case XII., § 37, provided the pulley is near the free end of the span. The superior limit in any of these cases is the inferior limit obtained in Case IX., § 25, and also in Case XII., § 36, provided the pulley lie near the shoulder end of the span.

43. Comparing the results in §§ 41, 42 with those obtained in Case X., § 27 (that is, with the case of a pulley on a shaft merely resting on a support at each end), we see that in the case of two equal spans the calculated speed for the pulley alone exceeds that in the case of a single span (equal in length to either of the two equal spans) in a certain ratio—that ratio depending on the position and size of the pulley.

Considering the superior limits in each case, the increase of speed due to the extra span is 10 per cent. when near the middle bearing, 24 (maximum advantage) when one third the span from the middle bearing, 19 at the centre of the span, and zero at the end bearing.

Considering the inferior limits in each case, the increase of speed is 35 per cent. when near the middle bearing, decreasing to 18 at the centre of the span and 8 per cent. near the end bearing.

44. *Experimental Results.* The same remarks apply here as in § 28, p. 310.

The following are the mean results of experiments made with different spans and with different positions of pulleys I. and II. (p. 285) on those spans. The shaft without the pulley has been investigated in §§ 15, 16, whilst the calculated speeds for the pulleys alone have been calculated from equation [A], § 39, p. 325, or, in the case of equal spans, from equation [B], § 39, p. 326.

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PULLEY I.

Number of experiment.	Date.	Conditions. l_1 in inches.	l_2 in inches.	c_1 in inches.	c_2 in inches.	*Observed speed.*	Calculated speed for shaft only.	Calculated speed for pulley only.	*Resulting calculated speed.*	*Percentage error.*
119	Dec. 20, 1892	2·91	29·10	28·10	1·00	1938	1943	19337	1933	+ ·2
120	,,	2·91	29·10	25·10	4·00	1798	1943	4375	1776	+ 1·2
121	,,	2·91	29·10	20·10	9·00	1522	1943	2583	1553	— 2·0
122	Jan. 30, 1893	2·91	26·10	14·55	14·55	1489	1943	2466	1526	— 2·5
123	,,	2·91	29 10	9·00	20·10	1651	1943	3418	1689	— 2·3
124	,,	2·91	29·10	4·00	25·10	1867	1943	8044	1889	— 1·2
125	,,	2·91	29·10	1·00	28·10	1935	1943	49951	1942	— ·3
126	Jan. 30, 1893	4·57	27·43	6·00	21·43	1975	2058	4900	1897	+ 3·9
127	,,	4·57	27·43	13·71	13·71	1675	2058	2601	1614	+ 3·6
128	,,	4·57	27·43	21·43	6·00	1789	2058	3346	1753	+ 2·0
129	,,	4·57	27·43	26·43	1·00	2029	2058	19756	2047	— ·9
114	Dec. 20, 1892	16·00	16·00	1·00	15·00	4430	4484	31664	4440	— ·9
115	,,	16·00	16·00	4·00	12·00	3930	4484	8114	3925	+ ·1
116	,,	16·00	16·00	8·00	8·00	3420	4484	4987	3334	+ 2·5
117	,,	16·00	16·00	12·00	4·00	3846	4484	6318	3657	+ 4·9
118	,,	16·00	16·00	15·00	1·00	4402	4484	24550	4411	— ·2

PULLEY II.

Number of experiment.	Date.	Conditions. l_1 in inches.	l_2 in inches.	c_1 in inches.	c_2 in inches.	*Observed speed.*	Calculated speed for shaft only.	Calculated speed for pulley only.	*Resulting calculated speed.*	*Percentage error.*
98	Dec. 2, 1892	2·91	29·10	1·00	28·00	1983	1943	37711	1940	+ 2·2
99	,,	2·91	29·10	4·00	25·10	1894	1943	5410	1829	+ 3·4
100	,,	2·91	29·10	9·00	20 10	1459	1943	2280	1479	— 1·3
101	,,	2·91	29·10	14·55	14·55	1234	1943	1644	1255	— 1·7
102	,,	2·91	29·10	20·10	9·00	1279	1943	1722	1290	— ·8
103	Dec. 5, 1892	2·91	29·10	25·10	4·00	1640	1943	2933	1620	+ 1·2
104	,,	2·91	29·10	28·00	1·00	1975	1943	15317	1928	+ 2·4
105	Dec. 5, 1892	4·57	27·43	26·43	1·00	2177	2058	15608	2040	+ 6·3
106	Dec. 7, 1892	4·57	27·43	21·43	6 00	1570	2058	2233	1513	+ 3·6
107	,,	4·57	27·43	13·71	13·71	1347	2058	1745	1330	+ 1·3
108	Dec. 8, 1892	4·57	27·43	6·00	21·43	1829	2058	3277	1743	+ 4·7
113	Dec. 19, 1892	16·00	16·00	1·00	15·00	4524	4484	24173	4411	+ 2·5
112	,,	16·00	16·00	4·00	12·00	3213	4484	4842	3286	— 2·2
109	Dec. 9, 1892	16·00	16·00	8·00	8·00	2600	4484	3325	2671	— 2·7
110	Dec. 14, 1892	16·00	16·00	12·00	4·00	3056	4484	4288	3100	— 1·4
111	Dec. 15, 1892	16·00	16·00	15·00	1·00	4220	4484	18816	4362	— 3·4

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These experiments show that the formula used for calculating the resulting speed —viz., $N_1N_2/\sqrt{(N_1^2 + N_2^2)}$, where N_1, N_2 are the separate speeds of whirl—holds, to a sufficient degree of approximation, whatever be the ratio of the spans or the position and size of the pulley. When one span is small compared to the other (Experiments 119–125 and 98–104), the conditions approximate to those required in Case XII., § 35. In this case the error is sometimes positive and sometimes negative, and the percentage error, with one exception, is under 3. The average error is −·1. In Experiments 105-108 and 126–129, in which the ratio of the spans is one-fifth, the error is practically positive throughout. The mean error is + 3. The calculated results could be made to approximate more closely to the actual speeds by using the formula $N_1N_2/\sqrt{(N_1^2 + \cdot 869\, N_2^2)}$ instead of $N_1N_2/\sqrt{(N_1^2 + N_2^2)}$, in which case the errors in Experiments 126, 127, 128, 129, 105, 106, 107, 108, would be − 2·0, −·5, − 3·0, − 8·2, ·4, 0·0, 2·8, 0·0 per cent. respectively, giving a mean error of − 1·3 per cent. But the speeds, as obtained by $N_1N_2/\sqrt{(N_1^2 + N_2^2)}$, are sufficiently near the actual speeds for practical purposes.

When the spans are equal (Experiments 109–118)—which is the most important case—one span being loaded, the error is sometimes positive and sometimes negative, but in only two cases (Experiments 111, 117) does it slightly exceed three per. cent. The mean error is −·1 per cent. The experiments, therefore, amply verify the theory.

45. The following are the mean results of experiments with the Pulleys I. and II. on the shaft at the same time. The spans have been each taken to be 16 inches. The first series include these experiments with Pulleys I. and II. on different spans, and the second with them on the same span—the positions in the two series being similar. The notation used to determine the position of the pulleys is the following:—a_1, a_2 are the distances of Pulley I. and c_1, c_2 of Pulley II. from the middle and outer bearings. The resulting calculated speed is taken to be $N_1N_2N_3/\sqrt{(N_1^2N_2^2 + N_2^2N_3^2 + N_3^2N_1^2)}$, where N_1, N_2, N_3 are the separate speeds of whirl for the shaft, Pulley I., and Pulley II. (see Case XVII., §§ 59–62; also §§ 30, 34). The calculated speed for the two pulleys neglecting the shaft is given, having been calculated from the formula $N_4 = N_2N_3/\sqrt{(N_2^2 + N_3^2)}$. For the three causes together the resulting speed is $N_1N_4/\sqrt{(N_1^2 + N_4^2)}$, which is equivalent to $N_1N_2N_3/\sqrt{(N_1^2N_2^2 + N_2^2N_3^2 + N_3^2N_1^2)}$.

PULLEYS I. and II.

Number of experiment.		Conditions.				Observed speed.		Calculated speed for shaft only (N_1).	Calculated speed for Pulley I. only (N_2).	Calculated speed for Pulley II. only (N_3).	Calculated speed for Pulleys I. and II. ($N_4 = N_2N_3/\sqrt{N_2^2 + N_3^2}$).	*Resulting calculated speed* ($N_1N_4/\sqrt{N_1^2 + N_4^2}$).	*Percentage error.*	
Pulleys on different spans.	Pulleys on same span.	a_1 in inches.	a_2 in inches.	c_1 in inches.	c_2 in inches.	Pulleys on different spans.	Pulleys on same span.						Pulleys on different spans.	Pulleys on same span.
130	149	12·00	4·00	12·00	4·00	2910	2683	4484	6318	4288	3548	2783	+4·4	−3·7
131	141	8·00	8·00	12·00	4·00	2877	2616	4484	4987	4288	3251	2632	+8·7	− ·6
132	142	4·00	12·00	12·00	4·00	2892	2720	4484	8114	4288	3791	2895	− ·1	−6·4
135	145	12·00	4·00	8·00	8·00	2583	2279	4484	6318	3325	2942	2460	+4·7	−8·0
134	144	8·00	8·00	8·00	8·00	2515	2215	4484	4987	3325	2766	2354	+6·4	−6·3
133	143	4·00	12·00	8·00	8·00	2562	2344	4484	8114	3325	3077	2537	+1·0	−8·2
136	146	12·00	4·00	4·00	12·00	2945	2816	4484	6318	4842	3843	2918	+ ·9	−3·6
137	147	8·00	8·00	4·00	12·00	2773	2635	4484	4987	4842	3474	2746	+1·0	−4·2
138	148	4·00	12·00	4·00	12·00	2929	2929	4484	8114	4842	4158	3049	−4·1	−4·1
139	..	1·00	15·00	1·00	15·00	4158	..	4484	31664	24173	19214	4367	−5·0	
140	..	15·00	1·00	15·00	1·00	4105	..	4484	24550	18816	14923	4295	−4·6	

The formula by which the resulting speed is calculated, viz. :—

$$N_1N_2N_3/\sqrt{(N_1^2N_2^2 + N_2^2N_3^2 + N_3^2N_1^2)},$$

gives, of course, the same calculated speed whether the pulleys be on different spans, or similarly placed on the same span. The experiments show that, with the pulleys on different spans, the observed speed is higher (with one exception) than when pulleys are similarly placed on the same span. In Experiments 138 and 148 the observed speed is the same in each case. Moreover, with the pulleys on different spans, the observed speed is, with one or two exceptions, in excess of the calculated speed; whilst, when on the same span it is, without exception, less than the calculated speed. In the former case, the average error is about + 3 per cent., and in the latter, about − 5 per cent., giving a mean of − 1 per cent. Either one or other of the separate errors (Experiments 130–140 or 141–149) could be reduced by the introduction of a constant in the denominator of the expression $N_1N_2N_3/\sqrt{(N_1^2N_2^2 + N_2^2N_3^2 + N_3^2N_1^2)}$, as in §§ 33, 34, but whilst reducing one it would also increase the other.

Considering, however, the complexity of the problem the preceding results justify, to a remarkable degree, the assumptions that have had to be made in the course of the investigation.

The experiments made with the pulleys on different spans are very instructive as showing how one pulley affects the other in regard to whirling. For example, Experiments 130, 131, 135, and 134 show that, when the two pulleys are both taken into account, the calculated speed is much too low. Hence we may infer that if Pulley I (which is the lighter of the two) be placed on the far side of the centre of its span from the middle bearing, its effect on the whirling speed is very small. The whirling speed may, in fact, be taken as that resulting from the combined effects of the heavier pulley and the shaft. On this assumption, the calculated whirling speeds in the above four experiments would be (see § 44, Experiments 110, 109) 3056, 3056, 2600 and 2600 respectively, and the percentage errors would be + 5, + 6·2, + 7, and + 3·4, instead of + 4·4, + 8·7, + 4·7, and + 6·4.

46. The discrepancies between the observed and calculated results are accounted for by the fact that the empirical formula—viz., $N_1N_2/\sqrt{(N_1^2 + N_2^2)}$—upon which the resulting calculated speeds are based, is not strictly accurate. In the case, however, of two or more equal spans with pulleys on each span, that formula gives calculated results *less* than the observed results and, therefore, erring on the right side. This is apparent from Experiment 130–140, but it might also be proved by considering the case of two equal spans with a pulley placed in the centre of each span. If the two pulleys be of the same size and weight they will have, separately, the same whirling speed. Let that whirling speed be N_1. Then, from § 41 or 42, we have

$$N_1 \propto 2\cdot900 \sqrt{(gEI/Wl^3)}, \text{ about},$$

where l = length of a single span. Using the ordinary formula, the resulting whirling speed, due to both pulleys, will be $N_1/\sqrt{2}$ (see § 62), so that the resulting whirling speed for the two pulleys will be proportional to

$$2 \cdot 05 \sqrt{(g\mathrm{EI}/\mathrm{W}l^3)}.$$

But, since the two spans are equal and similarly loaded, it is clear that there is no bending moment on the middle bearing. Consequently, as in the case of an unloaded shaft, § 15, the spans will whirl independently of each other, and the actual speed of whirl will therefore be proportional to

$$2 \cdot 45 \sqrt{(g\mathrm{EI}/\mathrm{W}l^3)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{see § 27}),$$

where E, I, W and l have the same values as before. Hence the whirling speed, as given by the formula used in the investigation, is only 84 per cent. of the actual whirling speed of the pulleys. When the shaft is also taken into account the difference between the two calculated will be decreased by an amount depending upon the relation between the whirling speed of the shaft, taken separately, and the whirling speeds for the pulleys as calculated above.

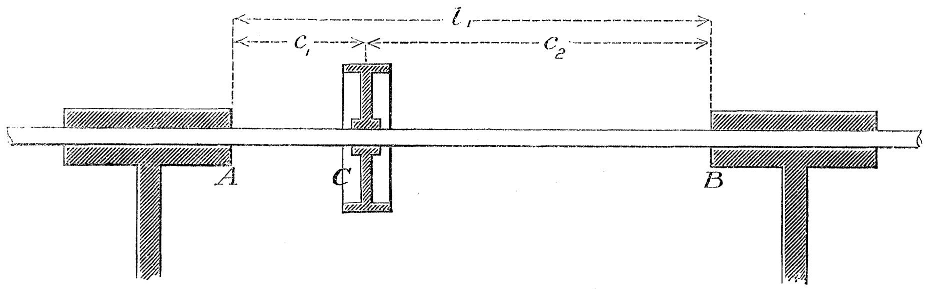
Reasoning in a similar way, we may conclude that when the spans are not similarly loaded, the whirling speeds as obtained in the investigation will be less than the actual whirling speeds. In other words, the formula used to determine the resulting speed of whirl errs on the right side.

Case XIV.

47. Shaft, length l, fixed in direction at each end and loaded with a pulley, weight W, and moment of inertia I′, at distances c_1, c_2 from the shoulders.

Thus—

Fig. 18.



Take the origin at A. Then from A to C we have (§ 21, p. 304, equation 2)

$$y = \frac{\mathrm{A}}{6} x^3 + \frac{\mathrm{B}}{2} x^2 + \mathrm{C}x + \mathrm{D}$$

and from C to B

$$y' = \frac{\mathrm{A}'}{6} x^3 + \frac{\mathrm{B}'}{2} x^2 + \mathrm{C}'x + \mathrm{D}'.$$

When $x = 0$,

$$y = 0, \quad dy/dx = 0;$$

whence

$$D = 0 \quad \ldots \ldots \quad (1),$$

$$C = 0 \quad \ldots \ldots \quad (2).$$

When $x = c_1$,

$$y = y', \quad dy/dx = dy'/dx;$$

whence

$$\tfrac{1}{6}(A - A')\,c_1^3 + \tfrac{1}{2}(B - B')\,c_1^2 + (C - C')\,c_1 + (D - D') = 0 \quad \ldots \quad (3),$$

$$\tfrac{1}{2}(A - A')\,c_1^2 + (B - B')\,c_1 + (C - C') = 0 \quad \ldots \quad (4).$$

When $x = l$,

$$y' = 0, \qquad dy'/dx = 0,$$

whence

$$\tfrac{1}{6}A'\,l^3 + \tfrac{1}{2}B'\,l^2 + C'l + D' = 0 \quad \ldots \ldots \quad (5),$$

$$\tfrac{1}{2}A'\,l^2 + B'l + C' = 0 \quad \ldots \ldots \quad (6).$$

When $x = c$ (at the pulley),

$$d\mathrm{L}/dx - d\mathrm{R}/dx = -\,\omega^2 y\,.\,\mathrm{W}/g \quad (\S\ 7,\ \text{equation } 5),$$

and

$$\mathrm{L} - \mathrm{R} = -\,\omega^2\mathrm{I}'\,dy/dx \quad (\S\ 7,\ \text{equation } 6),$$

whence we obtain, putting as before (§ 23, p. 305)

$$\alpha = \mathrm{W}\omega^2/g\mathrm{EI},\ \beta = \omega^2\mathrm{I}'/\mathrm{EI},\ \text{and}\ \beta = \alpha k^2,\ \text{where}\ k = \sqrt{(g\mathrm{I}'/\mathrm{W})},$$

$$A - A' = -\,\alpha\,\{\tfrac{1}{6}A\,c_1^3 + \tfrac{1}{2}B\,c_1^2 + Cc_1 + D\} \quad \ldots \ldots \quad (7),$$

$$(A - A')\,c_1 + (B - B') = -\,\beta\,\{\tfrac{1}{2}A\,c_1^2 + Bc_1 + C\} \quad \ldots \quad (8).$$

The elimination of the seven ratios

$$A : B : C : D : A' : B' : C' : D'$$

from the eight equations marked leads to

$$\alpha^2\,\tfrac{1}{12}\,k^2 c_1^4 c_2^4 + \alpha\,\{\tfrac{1}{3}\,l c_1^3 c_2^3 - k^2 c_1 c_2\,(c_1^3 + c_2^3)\} - l^4 = 0 \quad \ldots \quad [A],$$

a quadratic in ω^2 which is symmetrical with respect to c_1, c_2.

If $l = \infty$, then $c_2 = l$ and the equation reduces to

$$\alpha^2 \tfrac{1}{12} k^2 c_1^4 + \alpha \{\tfrac{1}{3} c_1^3 - k^2 c_1\} - 1 = 0,$$

the equation already obtained for the case of an overhanging shaft working in a shoulder (Case IX., § 23, p. 305).

From equation [A] we get

$$k^2 = \frac{4l}{\alpha c_1 c_2} \cdot \frac{l^3 - \alpha \frac{1}{3} c_1^3 c_2^3}{\frac{1}{3} \alpha c_1^3 c_2^3 - 4(c_1^3 + c_2^3)},$$

so that for whirling to be at all possible (see Case IV., § 24, p. 305), $\frac{1}{3} \alpha c_1^3 c_2^3$ must be $> l^3$ and $< 4(c_1^3 + c_2^3)$.

If $\alpha_1 c_1^3 c_2^3$ be equal to the first or second of these quantities, the corresponding value $g\omega$ gives the inferior or superior limit of the speed respectively. Moreover, *the period of whirl corresponding to the inferior limit of speed is identical with the natural period of vibration of the light shaft under the given conditions.*

The

$$\text{superior limit} = \text{inferior limit} \times 2 \cdot \sqrt{\left\{\left(\frac{c_1}{l}\right)^3 + \left(\frac{c_2}{l}\right)^3\right\}}.$$

Let

$$a = c_1/k, \qquad b = c_1/l\,;$$

that is, a and b are the ratios of the distance of the pulley from the nearer shoulder to the radius of gyration of the pulley, and to the whole span respectively.

Then the solution to equation [A], p. 337, may be expressed in the form

$$\alpha c_1^3 = 2\left[\left\{3\left(1 + \frac{b}{1-b}\right)^3 - \left(\frac{a^2}{1-b^2}\right)\right\} + \sqrt{\left\{3\left(1 + \frac{b}{1-b}\right)^3 - \left(\frac{a^2}{1-b^2}\right)\right\}^2 + \frac{3a^2}{(1-b)^4}}\,\right] \quad . \quad . \quad [B].$$

As in Case X. (§ 27), by assuming certain values for a and b, the corresponding values of αc_1^3 can be found, and so for any particular value of c_1 the value of ω readily calculated.

The following are the results obtained from equation [B]. The vertical columns give the values of θ for different values of a, the value of b being fixed; whilst the rows denote the values of θ for different values of b, the value of a being kept the same.

48. *Values of θ in the equation of $\omega = \theta\sqrt{(g\mathrm{EI}/wc^3)}$, c being the distance of the pulley from the nearer shoulder.*

		Values of $b = c/l$.					
		Very small.	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
Values of $a = c/k$.	Superior limit.	3·464	3·466	3·478	3·528	3·674	4·899
	·25	3·437	3·440	3·453	3·508	3·651	4·899
	·50	3·356	3·363	3·382	3·451	3·627	4·899
	·75	3·225	3·240	3·272	3·366	3·577	4·899
	1·00	3·048	3·082	3·135	3·266	3·521	4·899
	1·25	2·841	2·901	2·981	3·197	3·467	4·899
	1·50	2·628	2·824	2·856	3·076	3·419	4·899
	1·75	2·437	2·594	2·743	3·001	3·379	4·899
	2·10	2·282	2·479	2·653	2·941	3·346	4·899
	Inferior limit.	1·732	2·028	2·277	2·667	3·182	4·899

It will be noticed that when the span is very long and the pulley near the shoulder, so that c/l may be considered very small, the whirling speeds, for the same sized pulleys, are the same as those obtained in Cases IX. and X., §§ 25 and 36—the value of c being the same in all three cases.

The superior limit varies from twice the inferior (when the span is long and the pulley near the shoulder) to equality with it (when the pulley is at the centre of the span).

49. Comparing the results contained in the previous article with those under Case XII., §§ 36, 37 (that is with the case of a shaft working in a shoulder at one end and resting freely on a bearing at the other), we see that in the case where both ends

work in a shoulder the calculated speed for the pulley alone exceeds that in the case of a shaft free at one end and working in a shoulder at the other, in a certain ratio—that ratio depending on the position and size of the pulley.

Considering the superior limits in each case, the increase of speed due to the two shoulders is zero at the shoulder end, increasing to 27 per cent. at the centre of the span, and 100 per cent. at the free end. Considering the inferior limits in each case, the increase of speed due to the two shoulders is 2 per cent. near the shoulder end, increasing to 32 at the centre of the span, and 180 per cent. near the free end of the shaft.

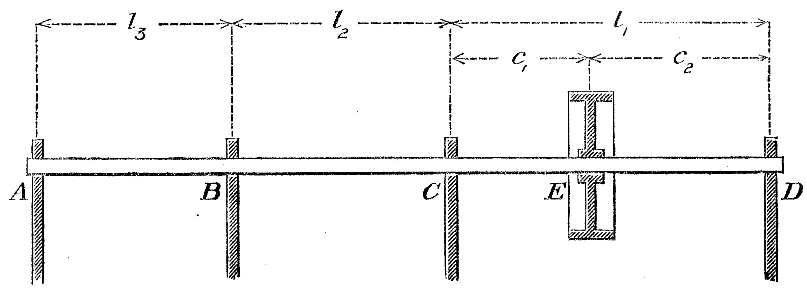
Again, comparing the results obtained in the present case with those obtained in Case X., § 27 (that is, with the case of a shaft merely resting on a bearing at each end) we see that, considering the superior limits in each case, the increase of speed due to the two shoulders is 100 per cent., whatever be the position of the pulley; whilst considering the inferior limits the increase of speed near the bearing is 233 per cent., decreasing to 100 at the centre.

Case XV.

50. SHAFT SUPPORTED ON FOUR BEARINGS, l_1, l_2, AND l_3 FEET APART RESPECTIVELY, AND LOADED WITH A PULLEY, WEIGHT W, AND MOMENT OF INERTIA I′, ON THE OUTER SPAN OF LENGTH l_1—THE PULLEY BEING DISTANT c_1 FROM THE INNER, AND c_2 FEET FROM THE OUTER BEARING.

Thus—

Fig. 19.



We have, taking the origin at the bearing, C (§ 21, equation 2).

$$y = \frac{A}{6}x^3 + \frac{B}{2}x^2 + Cx + D, \text{ from A to B},$$

$$y' = \frac{A'}{6}x^3 + \frac{B'}{2}x^2 + C'x + D', \text{ from B to C},$$

$$y'' = \frac{A''}{6}x^3 + \frac{B''}{2}x^2 + C''x + D'', \text{ from C to E},$$

and

$$y''' = \frac{A'''}{6}x^3 + \frac{B'''}{2}x^2 + C'''x + D''', \text{ from E to D}$$

When $x = 0$,

$$y' = 0, \quad y'' = 0, \quad dy'/dx = dy''/dx, \quad d^2y'/dx^2 = d^2y''/dx^2;$$

whence

$$D' = 0 \quad \ldots \quad (1),$$

$$D'' = 0 \quad \ldots \quad (2),$$

$$C' = C'' \quad \ldots \quad (3),$$

$$B' = B'' \quad \ldots \quad (4).$$

When $x = l_1$,

$$y''' = 0, \quad d^2y'''/dx^2 = 0;$$

whence

$$\frac{A'''}{6} l_1^3 + \frac{B'''}{2} l_1^2 + C''' l_1 + D''' = 0 \quad \ldots \quad (5),$$

$$A''' l_1 + B''' = 0 \quad \ldots \quad (6).$$

When $x = -(l_2 + l_3)$,

$$y = 0, \quad d^2y/dx^2 = 0;$$

whence

$$-\tfrac{1}{6}A(l_2 + l_3)^3 + \tfrac{1}{2}B(l_2 + l_3)^2 - C(l_2 + l_3) + D = 0 \quad \ldots \quad (7),$$

$$-A(l_2 + l_3) + B = 0 \quad \ldots \quad (8).$$

When $x = -l_2$,

$$y = 0, \quad y' = 0, \quad dy/dx = dy'/dx, \quad d^2y/dx^2 = d^2y'/dx^2;$$

whence

$$-\tfrac{1}{6}Al_2^3 + \tfrac{1}{2}Bl_2^2 - Cl_2 + D = 0 \quad \ldots \quad (9),$$

$$-\tfrac{1}{6}A'l_2^3 + \tfrac{1}{2}B'l_2^2 - C'l_2 + D' = 0 \quad \ldots \quad (10),$$

$$\tfrac{1}{2}(A - A')l_2^2 - (B - B')l_2 + (C - C') = 0 \quad \ldots \quad (11),$$

$$-(A - A')l_2 + (B - B') = 0 \quad \ldots \quad (12).$$

Again, at the pulley E, when $x = c_1$,

$$y'' = y''', \quad dy''/dx = dy'''/dx,$$

$$dL/dx - dR/dx = -W/g \,.\, \omega^2 y'' \quad (\S\ 7, \text{ equation } 5),$$

$$L - R = -\omega^2 I' \, dy''/dx \quad (\S\ 7, \text{ equation } 6);$$

whence, putting as before (§ 23, p. 305),

$$\alpha = W\omega^2/gEI, \ \beta = \omega^2 I'/EI, \text{ and therefore } \beta = \alpha k^2, \text{ where } k = \sqrt{(gI'/W)},$$

$$\frac{A'' - A'''}{6} c_1^3 + \frac{B'' - B'''}{2} c_1^2 + (C'' - C''') c_1 + (D'' - D''') = 0 \quad . \quad . \quad (13),$$

$$\frac{A'' - A'''}{2} c_1^2 + (B'' - B''') c_1 + (D'' - D''') = 0 \quad . \quad . \quad . \quad . \quad (14),$$

$$A'' - A''' = -\alpha \left\{ \frac{A''}{6} c_1^3 + \frac{B''}{2} c_1^2 + C'' c_1 + D'' \right\} \quad . \quad . \quad . \quad . \quad (15),$$

$$(A'' - A''') c_1 + (B'' - B''') = -\beta \left\{ \frac{A''}{2} c_1^2 + B'' c_1 + C'' \right\} \quad . \quad . \quad (16).$$

The elimination of the fifteen ratios

$$A : B : C : D : A' : B' : C' : D' : A'' : B'' : C'' : D'' : A''' : B''' : C''' : D''$$

from the sixteen equations marked leads to

$$\begin{aligned} &\alpha^2 . \tfrac{1}{3} k^2 c_1^3 c_2^3 \{(l_2 + l_3)(l_2 + c_1) + \tfrac{1}{3} l_2 l_3\} \\ &\quad + \alpha \Big[c_1^2 c_2^2 \Big\{ (l_2 + l_3) \Big(l_1 l_2 + c_1 . \overline{l_1 + \tfrac{c_2}{3}} \Big) + \tfrac{1}{3} l_1 l_2 l_3 \Big\} \\ &\quad - k^2 \{ (l_2 + l_3)(l_2 . \overline{c_1^3 + c_2^3} + c_1 . \overline{c_1^3 + 4c_2^3}) + \tfrac{1}{3} l_2 l_3 (c_1^3 + c_2^3) \} \Big] \\ &\quad - l_1^2 \{ (l_2 + l_3)(3l_2 + 4l_1) + l_2 l_3 \} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [A], \end{aligned}$$

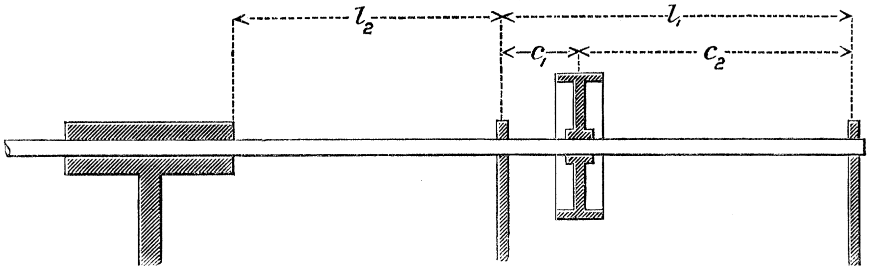
a quadratic in ω^2 which is not symmetrical with respect to c_1, c_2.

If in equation [A] we put $l_3 = 0$, it reduces to

$$\begin{aligned} &\alpha^2 . \frac{k^2 c_1^3 c_2^3}{9} (c_1 + l_2) \\ &\quad + \alpha [\tfrac{1}{3} c_1^2 c_2^2 \{ l_1 l_2 + c_1 (l_1 + \tfrac{1}{3} c_2) \} - k^2 \{ \tfrac{1}{3} c_1 (c_1^3 + 4c_2^3) + l_1 l_2 (\tfrac{1}{3} l_1^2 - c_1 c_2) \}] \\ &\quad - l_1^2 (l_2 + \tfrac{4}{3} l_1) = 0, \end{aligned}$$

which is identical with the equation obtained independently, but not reproduced here, of two spans, one of which is loaded, the outer end of the shaft on the loaded span merely resting on a bearing, whilst the outer end of the unloaded span works in a shoulder. Thus—

Fig. 20.



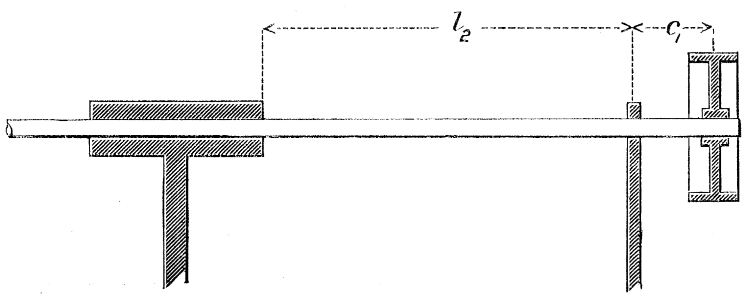
If, in addition, in the above equation we put (i) $l_2 = 0$ and (ii) ∞^y, it further reduces to the two equations already obtained in Case XII., § 35, p. 320, and Case X., § 26, p. 308.

By making l_1 and c_2 each equal to infinity, the equation further reduces to

$$\alpha^2 \tfrac{1}{3} k^2 c_1^{\,3} (l_2 + c_1) + \alpha \{c_1^{\,2} (l_2 + \tfrac{4}{3} c_1) - k^2 (l_2 + 4c_1)\} - 4 = 0,$$

which is the equation for a single span overhanging at one end and working in a shoulder at the other, the pulley being at the end of the overhanging portion. Thus

Fig. 21.



By putting, in this equation, $l_2 = 0$ we obtain the equation already obtained in Case IX., § 23, p. 305.

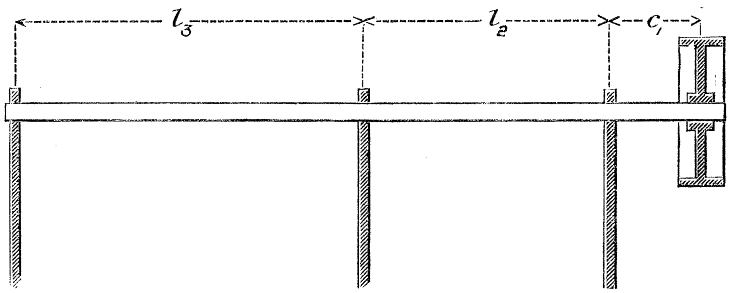
If in equation [A] we put $l_3 = \infty$, it immediately reduces to the equation already obtained for the case of a shaft of two spans, the shaft merely resting on the bearings at the ends, loaded with a pulley on one of the spans. (Case XIII., § 39, p. 325.)

Again, if in equation [A] we make l_1 and c_2 each equal to infinity, it reduces to

$$\alpha^2 \tfrac{1}{3} k^2 c_1^{\,3} \{(l_2 + l_3)(l_2 + c_1) + \tfrac{1}{3} l_2 l_3\} + \alpha [c_1^{\,2} \{(l_2 + l_3)(l_2 + \tfrac{4}{3} c_1) + \tfrac{1}{3} l_2 l_3\} - k^2 \{(l_2 + l_3)(l_2 + 4c_1) + \tfrac{1}{3} l_2 l_3\}] - 4 (l_2 + l_3) = 0,$$

which is the equation for an overhanging shaft loaded at the end, and having two spans on one side. That is, for the case of

Fig. 22.



By making $l_3 = \infty$ in the last equation, it becomes identical with that already investigated for the case of a single span overhanging on one side (Case XI., § 31, p. 313).

51. In the case of three spans, one of the end ones of which is loaded, the three cases which at once suggest themselves for full investigation are :—

(1). The two unloaded spans zero.

(2). The two unloaded spans infinite.

(3). All the three spans equal.

It has been shown that the first two cases have been already investigated (Cases XII. and X.). *It only remains to solve the third case when all the spans are equal.*

If

$$l_1 = l_2 = l_3 = l,$$

equation [A] on p. 341, becomes

$$\alpha^2 \tfrac{1}{3} k^2 c_1^3 c_2^3 (7l + 6c_1)$$

$$+ \alpha \{c_1^2 c_2^2 (7l^2 + 2c_1 . \overline{3c_1 + 4c_2}) - k^2 (7l . \overline{c_1^3 + c_2^3} + 6c_1 . \overline{c_1^3 + 4c_2^3})\} - 45l^3 = 0 \quad [B],$$

from which we immediately get

$$k^2 = \frac{1}{3\alpha c_1 c_2} \cdot \frac{15l^3 c_1 c_2 - \alpha \tfrac{1}{3} c_1^3 c_2^3 (7l^2 + 2c_1 . \overline{3c_1 + 4c_2})}{\alpha \tfrac{1}{3} c_1^3 c_2^3 (7l + 6c_1) - (7l . \overline{c_1^3 + c_2^3} + 6c_1 . \overline{c_1^3 + 4c_2^3})}$$

so that for whirling to be at all possible (§ 24, p. 305.)

$$\alpha \frac{c_1^3 c_2^3}{3} \quad \text{must be} > \quad \frac{15l^3 c_1 c_2}{7l^2 + 2c_1 (3c_1 + 4c_2)}$$

and

$$< \frac{7l (c_1^3 + c_2^3) + 6c_1 (c_1^3 + 4c_2^3)}{7l + 6c_1}$$

If $\alpha c_1^3 c_2^3/3$ be equal to the first or second of these expressions, the corresponding value of ω gives the inferior and superior limit of speed respectively. Moreover, *the period of whirl corresponding to the inferior limit of speed is identical with the natural period of vibration of the light shaft under the given conditions.*

The

$$\text{superior limit} = \text{inferior limit} \times \sqrt{\left(\frac{7l(c_1^3 + c_2^3) + 6c_1(c_1^3 + 4c_2^3)}{7l + 6c_1} \times \frac{7l^2 + 2c_1(3l + c_2)}{15l^3 c_1 c_2}\right)}.$$

Let

$$a_1 = c_1/k \quad \text{and} \quad b_1 = c_1/l;$$

that is, a_1 and b_1 are the ratios of the distance of the pulley from the inner bearing to the radius of gyration of the pulley and one of the spans respectively. Also, let a_2, b_2 be the corresponding ratios when the distance of the pulley is measured from the end bearing; that is,

$$a_2 = c_2/k \quad \text{and} \quad b_2 = c_2/l.$$

Then the solution to equation [B] may be put in either of the forms

$$\frac{2(7 + 6b_1) \cdot \alpha c_1^3}{3} = 7\left(1 + \frac{b_1^3}{(1 - b_1)^3}\right) + 6b_1\left(4 + \frac{b_1^3}{(1 - b_1)^3}\right) - a_1^2\left(\frac{7 + 8b_1 - 2b_1^2}{1 - b_1}\right)$$
$$+ \sqrt{\left[7\left(1 + \frac{b_1^3}{(1 - b_1)^3}\right) + 6b_1\left(1 + \frac{b_1^3}{(1 - b_1)^3}\right) - a_1^2\left(\frac{7 + 8b_1 + 2b_1^2}{1 - b_1}\right)\right]^2 + a_1^2\frac{60b_1(7 + 6b_1)}{(1 - b_1)^3}} \quad \text{[C.]}$$

$$\frac{2(13 - 6b_2) \cdot \alpha c_2^3}{3}$$
$$= 7\left(1 + \frac{b_2^3}{(1 - b_2)^3}\right) + 6(1 - b_2)\left(1 + 4\frac{b_2^3}{(1 - b_2)^3}\right) - a_2^2\left\{\frac{7}{1 - b_2} + 2(3 + b_2)\right\}$$
$$+ \sqrt{\left[7\left(1 + \frac{b_2^3}{(1 - b_2)^3}\right) + 6(1 - b_2)\left(1 + 4\frac{b_2^3}{(1 - b_2)^3}\right) - a_2^2\left\{\frac{7}{1 - b_2} + 2(3 + b_2)\right\}\right]^2 + a_2^2\frac{60b_2(13 - 6b_2)}{(1 - b_2)^3}} \quad \text{[D.]}$$

As in Cases X.–XIV. (§§ 27, 32, 36, 37, 41, 42), by assuming certain values for a_1, b_1, or a_2, b_2, the corresponding values of αc_1^3, or αc_2^3 can be found, and so, for any particular value of c_1 or c_2, the value of ω is readily calculated. Two sets of results have thus been compiled. The first set (obtained from equation [C]) gives the values of αc_1^3 for different values of a_1 and b_1, and is applicable when the pulley lies

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between the inner bearing and the centre of the span; whilst the second set (obtained from equation [D]) gives values of αc_2^3 for different values of a_2 and b_2, and is applicable when the pulley lies between the end bearing and the centre of the span.

52. *Values of* θ_1 *in the equation* $\omega = \theta_1\sqrt{(g\mathrm{EI}/\mathrm{W}c_1^3)}$ *when the pulley lies between the inner bearing and the centre of the span, and* c_1 = *distance of pulley from inner bearing.*

		Values of $b_1 = c_1/l$.					
		Very small	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
Values of $a_1 = c_1/k$.	Superior limit	1·732	1·927	2·037	2·168	2·318	2·950
	·25	1·677	1·883	1·998	2·137	2·298	2·949
	·50	1·500	1·749	1·887	2·052	2·243	2·948
	·75	1·146	1·540	1·724	1·939	2·174	2·946
	1·00	0	1·311	1·558	1·817	2·109	2·944
	1·25	do.	1·142	1·430	1·740	2·057	2·942
	1·50	do.	1·042	1·345	1·679	2·018	2·940
	1·75	do.	·984	1·291	1·636	1·990	2·937
	2·00	do.	·948	1·256	1·607	1·970	2·933
	Inferior limit	do.	·845	1·142	1·501	1·890	2·928

53. *Values of θ_2 in the equation $\omega = \theta_2\sqrt{(g\text{EI}/\text{W}c_2^3)}$ when the pulley lies between the outer bearing and the centre of the span, and c_2 = distance of pulley from outer bearing.*

		Values of $b_2 = c_2/l$.					
		Very small	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
Values of $a_2 = c_2/k$.	Superior limit.	1·732	1·735	1·748	1·800	1·945	2·950
	·25	1·677	1·683	1·701	1·767	1·930	2·949
	·50	1·500	1·525	1·569	1·681	1·892	2·948
	·75	1·146	1·275	1·388	1·578	1·850	2·946
	1·00	0	1·025	1·230	1·493	1·815	2·944
	1·25	do.	·876	1·129	1·436	1·790	2·942
	1·50	do.	·801	1·069	1·398	1·772	2·940
	1·75	do.	·759	1·033	1·373	1·760	2·937
	2·00	do.	·735	1·010	1·357	1·751	2·933
	Inferior limit	do.	·665	·938	1·317	1·717	2·906

The same pulley placed at equal distances from the inner and outer bearings whirls at different speeds, those nearer the inner bearing being considerably higher than those near the end bearing.

For further remarks, see those made in § 42, p. 329, which apply also to the present case.

54. Comparing these results (§§ 52, 53) with those obtained in Case XIII., §§ 41, 42, that is, with the case of two equal spans (one of which is loaded), we see that in the case of three equal spans, an outer span of which is loaded, the calculated speed for the pulley alone exceeds that in the case of only two spans in a certain ratio, that ratio depending on the position and size of the pulley.

Considering the superior limit in each case, the increase of speed due to the additional span of length equal to the length of either of the two spans (the two unloaded spans being on the same side of the loaded span) is 1·1 per cent. when near the inner bearing, 1·9 when one-third the span from the inner bearing (maximum advantage), 1·4 at the centre of the span, and zero at the outer bearing. Considering the inferior limits in each case, the increase of speed is 2·9 per cent. when near the inner bearing, decreasing to 1·3 at the centre of the span and ·6 per cent. at the end bearing.

We thus see that, in the present case, the effect of the second unloaded span from the loaded one, in increasing the speed at which the pulley will cause the shaft to whirl, can never be such as to cause the increase in the whirling speed to exceed 3 *per cent. of that calculated on the assumption that the effect of that second unloaded span is altogether neglected.*

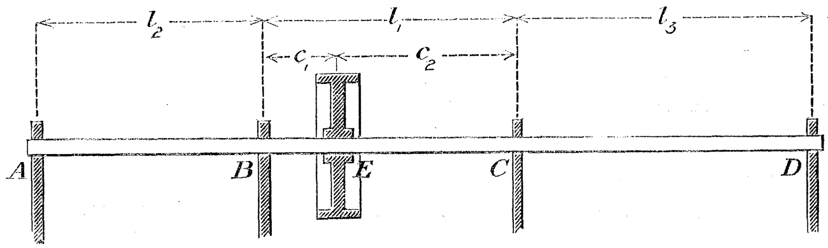
When the effect of the shaft is also taken into account, the increase in the whirling speed due to the third span will be less than 3 per cent. (§ 62).

Case XVI.

55. SHAFT SUPPORTED ON FOUR BEARINGS, l_2, l_1, AND l_3 FEET APART RESPECTIVELY, AND LOADED WITH A PULLEY, WEIGHT W, AND MOMENT OF INERTIA I′, ON THE MIDDLE SPAN OF LENGTH l_1—THE PULLEY BEING DISTANT c_1, c_2 FEET FROM THE BEARINGS.

Thus—

Fig. 23.



We have, taking the origin at the bearing B (§ 21, equation 2),

$$y = \frac{A}{6} x^3 + \frac{\ }{2} x^2 + Cx + D\text{, from A to B,}$$

$$y' = \frac{A'}{6} x^3 + \frac{B'}{2} x^2 + C'x + D'\text{, from B to E,}$$

$$y'' = \frac{A''}{6} x^3 + \frac{B''}{2} x^2 + C''x + D''\text{, from E to C,}$$

$$y''' = \frac{A'''}{6} x^3 + \frac{B'''}{2} x^2 + C'''x + D'''\text{, from C to D.}$$

When $x = 0$,

$$y = 0, \qquad y' = 0,$$
$$dy/dx = dy'/dx, \qquad d^2y/dx^2 = d^2y'/dx^2\,;$$

whence

$$D = 0 \quad \ldots \ldots \quad (1);$$

$$D' = 0 \quad \ldots \ldots \quad (2),$$

$$C = C' \quad \ldots \ldots \quad (3),$$

$$B = B' \quad \ldots \ldots \quad (4).$$

When $x = -l_2$,

$$y = 0, \qquad d^2y/dx^2 = 0\,;$$

whence

$$-\tfrac{1}{6}\,Al_2^3 + \tfrac{1}{2}\,B\,l_2^2 - Cl_2 + D = 0 \quad \ldots \ldots \quad (5),$$

$$-Al_2 + B = 0 \quad \ldots \ldots \quad (6).$$

When $x = l_1$,

$$y'' = 0, \qquad y''' = 0,$$
$$dy''/dx = dy'''/dx, \qquad d^2y''/dx^2 = d^2y'''/dx^2\,;$$

whence

$$\tfrac{1}{6}\,A''l_1^3 + \tfrac{1}{2}\,B''l_1^2 + C''l_1 + D'' = 0 \quad \ldots \ldots \quad (7),$$

$$\tfrac{1}{6}\,A'''l_1^3 + \tfrac{1}{2}\,B'''l_1^2 + C'''l_1 + D''' = 0 \quad \ldots \ldots \quad (8),$$

$$\tfrac{1}{2}\,(A'' - A''')\,l_1^2 + (B'' - B''')\,l_1 + (C'' - C''') = 0 \quad \ldots \ldots \quad (9),$$

$$(A'' - A''')\,l_1 + (B'' - B''') = 0 \quad \ldots \ldots \quad (10).$$

When $x = l_1 + l_3$,

$$y''' = 0, \qquad d^2y'''/dx^2 = 0\,;$$

whence

$$\tfrac{1}{6}\,A'''\,(l_1 + l_3)^3 + \tfrac{1}{2}\,B'''\,(l_1 + l_3)^2 + C'''\,(l_1 + l_3) + D''' = 0 \quad \ldots \quad (11),$$

$$A'''\,(l_1 + l_3) + B''' = 0 \quad \ldots \ldots \quad (12).$$

Again, at the pulley E, when $x = c_1$,

$$y' = y'', \qquad dy'/dx = dy''/dx,$$
$$dL/dx - dR/dx = -W/g\,.\,\omega^2 y' \quad (\S\,7,\ \text{equation } 5),$$
$$L - R = -\omega^2 I'\,dy'/dx \quad (\S\,7,\ \text{equation } 6)\,;$$

whence

$$\tfrac{1}{6}(A' - A'')c_1^3 + \tfrac{1}{2}(B' - B'')c_1^2 + (C' - C'')c_1 + (D' - D'') = 0 \quad . \quad (13),$$

$$\tfrac{1}{2}(A' - A'')c_1^2 + (B' - B'')c_1 + (C' - C'') = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (14),$$

$$A' - A'' = -\alpha(\tfrac{1}{6}A'c_1^3 + \tfrac{1}{2}B'c_1^2 + C'c_1 + D') \quad . \; . \; . \; . \; . \; . \; . \; . \quad (15),$$

$$(A' - A'')c_1 + (B' - B'') = -\beta(\tfrac{1}{2}A'c_1^2 + B'c_1 + C') \quad . \; . \; . \; . \; . \; . \quad (16),$$

where, as in § 23, p. 305,

$$\alpha = W\omega^2/gEI, \qquad \beta = \omega^2 I'/EI, \qquad \text{and } \beta = \alpha k^2, \text{ where } k = \sqrt{(gI'/W)}.$$

The elimination of the fifteen ratios

$$A : B : C : D : A' : B' : C' : D' : A'' : B'' : C'' : D'' : A''' : B''' : C''' : D'''$$

from the sixteen equations marked leads to

$$\begin{aligned}\alpha^2 \, . \, \tfrac{1}{36}k^2c_1^3c_2^3\{12(l_2c_2 + l_3c_1) + 9c_1c_2 + 16l_2l_3\} \\ + \alpha[\tfrac{1}{3}c_1^2c_2^2\{3l_1(c_1c_2 + l_3c_1 + l_2c_2) + c_1c_2(l_2 + l_3) + 4l_1l_2l_3\} \\ - \tfrac{1}{3}k^2\{9c_1c_2(c_1^3 + c_2^3) + 3l_2c_2(c_2^3 + 4c_1^3) + 3l_3c_1(c_1^3 + 4c_2^3) + 4l_2l_3(c_1^3 + c_2^3)\}] \\ - l_1^2\{3l_1^2 + 4(l_2l_3 + l_3l_1 + l_1l_2)\} = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad [A],\end{aligned}$$

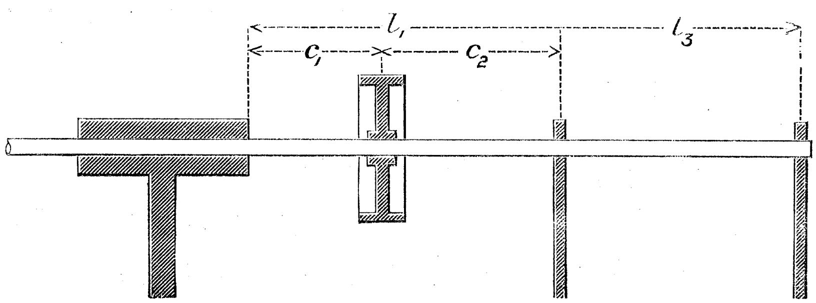
a quadratic in ω^2, symmetrical with respect to (l_2, c_1) and (l_3, c_2).

If in equation [A] we put $l_2 = 0$, it reduces to

$$\begin{aligned}\alpha^2 \, . \, \tfrac{1}{6}k^2c_1^4c_2^3(\tfrac{1}{2}c_2 + \tfrac{2}{3}l_3) \\ + \alpha \, . \, [\tfrac{1}{3}c_1^3c_2^2\{c_2l_1 + l_3(l_1 + \tfrac{1}{3}c_2)\} - c_1k^2\{l_1c_2(l_1^2 - 3c_1c_2) + \tfrac{1}{3}l_3(c_1^3 + 4c_2^3)\}] \\ - l_1^3(l_1 + \tfrac{4}{3}l_3) = 0,\end{aligned}$$

which is identical with the equation obtained independently, but not reproduced here, of two spans, one of which is loaded, the outer end of the shaft on the loaded span working in a shoulder, whilst that of the unloaded span merely rests on the support. Thus

Fig. 24.



If we further put $l_3 = 0$, the equation reduces to that already obtained for the case of a shaft working in a shoulder at each end (Case XIV., § 47).

If $l_3 = \infty^y$, instead of 0, we obtain the equation for the case of a shaft working in a sleeve at one end and merely resting on a bearing at the other (Case XII., § 35).

If in equation [A] we put $l_3 = \infty^y$, it reduces to that already obtained for two spans, one of which is loaded (Case XIII., § 39). If in addition $l_2 = \infty$ we obtain Case X., § 26.

56. In the case of three spans, the middle one of which is loaded, the three cases which at once suggest themselves for full investigation are—

(1.) the two unloaded spans zero,
(2.) the two unloaded spans infinite,
(3.) all three spans equal.

It has been shown that the first two cases have been already investigated (Cases XIV. and X.). *It only remains to solve the third case when all the spans are equal.*

If $l_2 = l_1 = l_3 = l$, equation [A] reduces to

$$\alpha^2 \cdot \tfrac{1}{36} k^2 c_1^3 c_2^3 (28 l^2 + 9 c_1 c_2) + \alpha \left[\tfrac{1}{3} c_1^2 c_2^2 l (7 l^2 + 5 c_1 c_2) - \tfrac{1}{3} k^2 \{(9 c_1 c_2 + 7 l^2)(c_1^3 + c_2^3) + 9 l c_1 c_2 (c_1^2 + c_2^2)\}\right] - 15 l^4 = 0 \quad \text{[B]},$$

from which we get

$$k^2 = \frac{12 l}{\alpha c_1 c_2} \cdot \frac{15 l_1^3 c_1 c_2 - \frac{1}{3} \alpha c_1^3 c_2^3 (7 l^2 + 5 c_1 c_2)}{\frac{1}{3} \alpha c_1^3 c_2^3 (28 l^2 + 9 c_1 c_2) - 4 \{(9 c_1 c_2 + 7 l^2)(c_1^3 + c_2^3) + 9 l c_1 c_2 (c_1^2 + c_2^2)\}},$$

so that for whirling to be at all possible we must have (see Case IX., § 24, p. 305),

$$\tfrac{1}{3} \alpha c_1^3 c_2^3 > \frac{15 l^3 c_1 c_2}{7 l^2 + 5 c_1 c_2}$$

and

$$< 4 \cdot \frac{(9 c_1 c_2 + 7 l^2)(c_1^3 + c_2^3) + 9 l c_1 c_2 (c_1^2 + c_2^2)}{28 l^2 + 9 c_1 c_2}.$$

If $\alpha c_1^3 c_2^3/3$ be equal to the first or second of these expressions, the corresponding value of ω gives the inferior or superior limit of the speed respectively. Moreover, *the period of whirl corresponding to the inferior limit of speed is identical with natural period of vibration of the light shaft under the given conditions.*

The

superior limit

$$= 2 \times \text{inferior limit} \times \sqrt{\left(\frac{(9 c_1 c_2 + 7 l^2)(c_1^3 + c_2^3) + 9 l c_1 c_2 (c_1^2 + c_2^2)}{28 l^2 + 9 c_1 c_2} \times \frac{7 l^2 + 5 c_1 c_2}{15 l^3 c_1 c_2}\right)}.$$

Let

$$a = c_1/k \text{ and } b = c_1/l,$$

that is, a and b are the ratios of the distance of the pulley from the nearest bearing to the radius of gyration of the pulley and to the span respectively.

Then the solution to equation [B] p. 350, may be expressed in the form

$$\alpha c_1^3 = \frac{6}{28 + 9b(1-b)}\left[\left\{\left(1+\frac{b}{1-b}\Big|^3\right)(7 + 9b\,.\,\overline{1-b}) + 9b\left(1+\frac{b}{1-b}\Big|^2\right)\right\} - a^2\left\{\frac{7}{1-b} + 5b\right\}\right.$$

$$+ \sqrt{\left\{\left(1+\frac{b}{1-b}\Big|^3\right)(7 + 9b\,.\,\overline{1-b}) + 9b\left(1+\frac{b}{1-b}\Big|^2\right)\right\} - a^2\left\{\frac{7}{1-b} + 5b\right\}^2}$$

$$\left. + a^2\,\frac{15b(28 + 9b\,\overline{1-b})}{(1-b)^3}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad [C].$$

As in Cases X. and XIV. (§§ 27, 48), by assuming certain values for a and b, the corresponding values of αc_1^3 can be found and so, for any particular value of c_1, the value of ω readily calculated.

The following are the results obtained from equation [C]. The vertical columns give the value of θ for different values of a, the value of b being fixed; whilst the rows denote the value of θ for different values of b, the value of a being fixed.

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57. *Values of θ in the equation $\omega = \theta \sqrt{(g\text{EI}/\text{W}c^3)}$; c being the distance of the pulley from the nearer bearing.*

		Values of $b = c/l$.					
		Very small	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
Values of $a = c/k$.	Superior limit	1·732	1·907	2·013	2·157	2·356	3·303
	·25	1·677	1·862	1·975	2·129	2·340	3·303
	·50	1·500	1·729	1·867	2·055	2·300	3·303
	·75	1·146	1·523	1·714	1·957	2·250	3·303
	1·00	0	1·304	1·563	1·864	2·203	3.303
	1·25	do.	1·145	1·451	1·792	2·166	3·303
	1·50	do.	1·052	1·372	1·741	2·138	3·303
	1·75	do.	·997	1·323	1·705	2·117	3·303
	2·00	do.	·963	1·291	1·680	2·102	3·303
	Inferior limit	do.	·863	1·185	1·587	2·040	3·303

The superior limit thus varies from 2·21 times the inferior limit (when the pulley is near the bearing) to equality with it (at the centre of the span).

Moreover, when the span is very long, and the pulley near the bearing, so that c/l may be taken to be very small, no whirling can take place provided the radius of gyration is less that the distance of the pulley from the bearing. (See also §§ 27, 32, 37, 41, 42, 52, 53.)

58. Comparing these results with those obtained in Case X., § 27 (that is, with the case of a single span), we see that in the case of three equal spans, the middle one of which is loaded, the calculated speed for the pulley alone exceeds that in the case of a single span in a certain ratio—that ratio depending on the position and size of the

pulley. Considering the superior limits in each case, we see that the increase of speed due to two additional spans (each equal in length to the first span), one on each side, is, as regards the superior limits, 10 per cent. near the bearing, and 34·4 per cent. at the centre of the span; and, as regards the inferior limits, it is 41·7 per cent. near the bearing, and 34·4 per cent. at the centre of the span.

Again, comparing the case under discussion with Case XIV., § 48, in which the two end spans are zero (that is, the shaft works in a shoulder at each end), we see that the increase of speed, due to the two shoulders, is, as regards the superior limits, 100 per cent. near the bearing, and 48·3 per cent. at the centre of the span; and, as regards the inferior limits, 135 per cent. near the bearing, and 48·3 per cent. at the centre of the span.

Comparing the present case with the results obtained in Case XIII., §§ 41, 42 (that is, with the case of two equal spans, one of which is loaded), we see that the increase of speed due to the extra span (the loaded span being in the middle) is, as regards the superior limit, zero when the pulley is near the inner bearing, increasing to 13·6 per cent. at the centre, 21·6 (maximum advantage) at one-third the length of the span from the free end, and decreasing to 10 per cent. when near the free end; and, as regards the inferior limits, the increase of speed is 5 per cent. when near the middle bearing, 14·3 at the centre of the span, and 30·5 per cent. at the outer bearing.

Finally, comparing the results in the present case with those obtained in Case XV., §§ 52, 53 (that is, with the case of three equal spans, one of the end ones being loaded), it will be noticed that the results in the latter case are the higher when the pulley is near the bearing, as regards the superior limits, but less as regards the inferior limits. By further referring to what was proved in § 54, p. 346, *we may infer that if, in the present case, an additional equal span be added on each side (making, in all, five spans, the middle one being loaded), the effect of those additional spans, in increasing the speed at which the pulley will cause the shaft to whirl, will never be such as to cause the increase in the whirling speed to exceed one or two per cent. of that calculated on the assumption that the effects of the two additional spans are altogether neglected.*

When the effect of the shaft is also taken into account, the increase in the whirling speed, due to the two additional spans, will be still further reduced (§ 62).

This result, and that obtained in § 54, are extremely important, as they practically limit any problem to the case of three spans. In other words, in the case of a continuous shaft, supported on bearings, placed at equal distances apart, and loaded with a pulley on one of the spans, the whirling speed, due to that pulley, obtained by considering the loaded span, and the span or spans immediately adjacent to it on either side, is sufficiently accurate for practical purposes.

Case of two or more Pulleys.

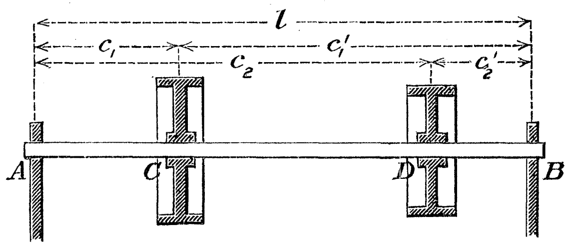
59. So far (Cases IX.–XVI., §§ 23–58) we have only fully investigated the effect of one pulley on a shaft supported in different ways, the effect of the shaft being neglected. It was shown in §§ 19, 20 that, even in simple cases, the equations obtained by considering the shaft and a single pulley together were too complicated to allow of a solution in a form convenient for practical application. The following case will show that, even in the simple case of a shaft freely supported at the two ends, the equations obtained by considering the effect of the two pulleys together—the effect of the shaft being altogether neglected—are also too complicated to allow of a solution which can be readily applied to any actual case.

Case XVII.

60. Shaft, length l, merely resting on a support at each end and loaded with two pulleys, weights W_1, W_2, and moments of inertia I_1, I_2, placed at distances c_1, c_1' and c_2, c_2' respectively, from the bearings.

Thus—

Fig. 25.



Taking the origin at the bearing A, we have (§ 21, equation 2)

$$y = \frac{A}{6}x^3 + \frac{B}{2}x^2 + Cx + D, \text{ from A to C},$$

$$y' = \frac{A'}{6}x^3 + \frac{B'}{2}x^2 + C'x + D', \text{ from C to D},$$

$$y'' = \frac{A''}{6}x^3 + \frac{B''}{2}x^2 + C''x + D'', \text{ from D to B}.$$

When

$$x = 0, \qquad y = 0, \qquad d^2y/dx^2 = 0;$$

whence

$$D = 0 \quad \text{(1)},$$

$$B = 0 \quad \text{(2)}.$$

When $x = c_1$,

$$y = y', \qquad dy/dx = dy'/dx,$$

$$d\mathrm{L}/dx - d\mathrm{R}/dx = - \mathrm{W}_1/g \,.\, \omega^2 y \quad (\S\, 7, \text{ equation } (5)),$$

$$\mathrm{L} - \mathrm{R} = - \omega^2 \mathrm{I}_1\, dy/dx \quad (\S\, 7, \text{ equation } (6));$$

whence,

$$\frac{\mathrm{A} - \mathrm{A}'}{6} c_1^3 + \frac{\mathrm{B} - \mathrm{B}'}{2} c_1^2 + (\mathrm{C} - \mathrm{C}') c_1 + (\mathrm{D} - \mathrm{D}') = 0 \quad . \quad . \quad . \quad . \quad (3),$$

$$\frac{\mathrm{A} - \mathrm{A}'}{2} c_1^2 + (\mathrm{B} - \mathrm{B}') c_1 + (\mathrm{C} - \mathrm{C}') = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

$$\mathrm{A} - \mathrm{A}' = - \frac{\mathrm{W}_1 \omega^2}{g\mathrm{EI}} \left(\mathrm{A}\, \frac{c_1^3}{6} + \mathrm{B}\, \frac{c_1^2}{2} + \mathrm{C} c_1 + \mathrm{D} \right) \quad . \quad . \quad . \quad . \quad (5),$$

$$(\mathrm{A} - \mathrm{A}') c_1 + (\mathrm{B} - \mathrm{B}') = - \frac{\omega^2 \mathrm{I}_1}{\mathrm{EI}} \left(\frac{\mathrm{A}}{2} c_1^2 + \mathrm{B} c_1 + \mathrm{C} \right) \quad . \quad . \quad . \quad . \quad (6).$$

Similarly, when $x = c_2$, we have

$$\frac{\mathrm{A}' - \mathrm{A}''}{6} c_2^3 + \frac{\mathrm{B}' - \mathrm{B}''}{2} c_2^2 + (\mathrm{C}' - \mathrm{C}'') c_2 + (\mathrm{D}' - \mathrm{D}'') = 0 \quad . \quad . \quad . \quad (7),$$

$$\frac{\mathrm{A}' - \mathrm{A}''}{2} c_2^2 + (\mathrm{B}' - \mathrm{B}'') c_2 + (\mathrm{C}' - \mathrm{C}'') = 0 \quad . \quad . \quad . \quad . \quad (8),$$

$$\mathrm{A}' - \mathrm{A}'' = - \frac{\mathrm{W}_2 \omega^2}{g\mathrm{EI}} \left(\mathrm{A}'\, \frac{c_2^3}{6} + \mathrm{B}'\, \frac{c_2^2}{2} + \mathrm{C}' c_2 + \mathrm{D}' \right) \quad . \quad . \quad . \quad . \quad (9),$$

$$(\mathrm{A}' - \mathrm{A}'') c_1 + (\mathrm{B}' - \mathrm{B}'') = - \frac{\omega^2 \mathrm{I}_2}{\mathrm{EI}} \left(\frac{\mathrm{A}'}{2} c_2^2 + \mathrm{B}' c_2 + \mathrm{C}' \right) \quad . \quad . \quad . \quad (10).$$

Again, when $x = l$,

$$y_2'' = 0, \qquad dy''/dx^2 = 0,$$

whence

$$\frac{\mathrm{A}''}{6} l^3 + \frac{\mathrm{B}''}{2} l^2 + \mathrm{C}'' l + \mathrm{D}'' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

$$\mathrm{A}'' l + \mathrm{B}'' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

Putting

$$\alpha_1 = \mathrm{W}_1 \omega^2 / g\mathrm{EI}, \quad \beta_1 = \omega^2 \mathrm{I}_1 / \mathrm{EI} \quad \text{and} \quad \beta_1 = \alpha_1 k_1^2, \quad \text{where} \quad k_1 = \sqrt{(g\mathrm{I}_1 / \mathrm{W}_1)},$$

$$\alpha_2 = \mathrm{W}_2 \omega^2 / g\mathrm{EI}, \quad \beta_2 = \omega^2 \mathrm{I}_2 / \mathrm{EI} \quad \text{and} \quad \beta_2 = \alpha_2 k_2^2, \quad \text{where} \quad k_2 = \sqrt{(g\mathrm{I}_2 / \mathrm{W}_2)},$$

the elimination of the eleven ratios

$$A : B : C : D : A' : B' : C' : D' : A'' : B'' : C'' : D''$$

leads to

$$\left\{ l + \alpha_1 \frac{c_1{}^3 c_1{}'}{6} + \alpha_2 \frac{c_2{}^3 c_2{}'}{6} + \beta_1 \frac{c_1{}^2}{2} + \beta_2 \frac{c_2{}^2}{2} + \beta_1\beta_2 \frac{c_1{}^2 l'}{2} + \alpha_1\alpha_2 \frac{c_1{}^3 c_2{}' l'^3}{36} \right.$$
$$\left. + \beta_1\alpha_2 \frac{c_1{}^2 c_2{}' l'^2}{4} + \alpha_1\beta_2 \frac{c_1{}^3 l'^2}{12} \right\}$$
$$\left\{ l + \alpha_1 \frac{c_1 c_1{}'^3}{6} + \alpha_2 \frac{c_2 c_2{}'^3}{6} + \beta_1 \frac{c_1{}'^2}{2} + \beta_2 \frac{c_2{}'^2}{2} + \beta_1\beta_2 \frac{l' c_2{}'^2}{2} + \alpha_1\alpha_2 \frac{c_1 c_2{}'^3 l'^3}{36} \right.$$
$$\left. + \alpha_1\beta_2 \frac{c_1 c_2{}'^2 l'^2}{4} + \beta_1\alpha_2 \frac{c_2{}'^3 l'^2}{12} \right\}$$
$$- \left\{ \alpha_1 c_1 c_1{}' + \alpha_2 c_2 c_2{}' + \beta_1 + \beta_2 + \beta_1\beta_2 l' + \alpha_1\alpha_2 \frac{c_1 c_2{}' l'^3}{6} + \beta_1\alpha_2 \frac{c_2{}' l'^2}{2} + \alpha_1\beta_2 \frac{c_1 l'^2}{2} \right\}$$
$$\left\{ \frac{l^3}{6} + \alpha_1 \frac{c_1{}^3 c_1{}'^3}{36} + \alpha_2 \frac{c_2{}^3 c_2{}'^3}{36} + \beta_1 \frac{c_1{}^2 c_1{}'^2}{4} + \beta_2 \frac{c_2{}^2 c_2{}'^2}{4} + \alpha_1\alpha_2 \frac{c_1{}^3 c_2{}'^3 l'^3}{216} + \beta_1\beta_2 \frac{c_1{}^2 c_2{}'^2 l'}{4} \right.$$
$$\left. + \beta_1\alpha_2 \frac{c_1{}^2 c_2{}'^3 l'^2}{24} + \alpha_1\beta_2 \frac{c_1{}^3 c_2{}'^2 l'^2}{24} \right\} = 0 \quad \ldots\ldots\ldots\ldots \quad [A],$$

in which

$$c_1 + c_1{}' = c_2 + c_2{}' = l,$$
$$c_2 - c_1 = c_1{}' - c_2{}' = l'.$$

If the second pulley be supposed removed, that is, if we put W_2 and I_2 each equal to zero in equation [A], we get

$$\omega^4 \frac{W_1 I_1}{9gE^2I^2} c_1{}^3 c_1{}'^3 + \omega^2 \left\{ \frac{W_1}{3gEI} l c_1{}^2 c_1{}'^2 + \frac{I_1 l}{3EI} (3c_1 c_1{}' - l^2) \right\} - l^3 = 0,$$

a result, of course, identical with that already obtained (Case X, § 26, p. 308).

It will be seen at once that the equation [A] is practically useless unless some special relation be assumed between the dimensions of the pulleys, &c., and even then it would be impossible to compile a table which could be used except in very few cases.

Cases, other than the above, in which a shaft is supported in a certain manner and carries two pulleys (for example, a span with an overhanging portion on one side and supporting one pulley between the bearings and another at the end of the overhanging portion) have been investigated, and in each case the result obtained was too complicated to admit of any practical assumption.

61. *The only alternative method is to consider the effects of the shaft (whatever be its mode of support) and each of the pulleys (whatever be their number, position, and size) separately, and so obtain the whirling speed for each on the assumption that all the others are neglected. By means of an empirical formula the whirling speed, when the effects of the shaft and of all the pulleys are taken into account, may be calculated from the separately calculated whirling speeds.*

62. The particular form of the empirical formula was found as follows :—

If a weight W_1 be supported by a spring which requires ϵ pounds to stretch it one foot, then the number of vibrations which that weight makes per second is

$$N_1 = \sqrt{(g\epsilon/W_1)}.$$

The number of vibrations which a second weight W_2 (attached at the same point of the spring as the weight W_1) makes is

$$N_2 = \sqrt{(g\epsilon/W_2)}\,;$$

and the number which the combined weight $(W_1 + W_2)$ would make is

$$N = \sqrt{\left\{\frac{g\epsilon}{W_1 + W_2}\right\}} = \frac{1}{\sqrt{(N_1^{-2} + N_2^{-2})}} = \frac{N_1 N_2}{\sqrt{(N_1^2 + N_2^2)}}.$$

In the same manner this formula would be strictly accurate in the case of a rod, however supported, provided that any concentrated loads which it might carry could be supposed concentrated at the same point. For example, if three loads be concentrated at the same point of the rod (the effect of the rod being neglected), and if the number of vibrations which each makes per second, when assumed independent of the others, be N_1, N_2, N_3, then the number of vibrations of the three together will be

$$\frac{N_1 N_2 N_3}{\sqrt{(N_2^2 N_3^2 + N_3^2 N_1^2 + N_1^2 N_2^2)}},$$

and so on for any number of loads.

If, however, the loads be concentrated at different points, the above formula will not be strictly true; for, in addition to the number of vibrations varying inversely as the square root of the weight, the value of ϵ will vary with some function of the distance of the weight from the point of support.

In the same manner, in the whirling of shafts, if N_1, N_2, N_3 be the whirling speeds due to three pulleys when each is considered independently of the remaining two, we have (§§ 25, 27, 32, 36, 37, 41, 42, 48, 52, 53 and 57), since $\omega \propto \theta\sqrt{(I/Wc^3)}$ and therefore as $d^2/\sqrt{(Wc^3)}$, where d = diameter of shaft,

$$N_1 = \phi_1 \frac{d^2}{\sqrt{(W_1 c_1^3)}}, \quad N_2 = \phi_2 \frac{d^2}{\sqrt{(W_2 c_2^3)}}, \quad N_3 = \phi_3 \frac{d^2}{\sqrt{(W_3 c_3^3)}},$$

where ϕ_1, ϕ_2, ϕ_3 are constants depending on the position and size of the pulleys ; and (assuming all the pulleys to be so near together that each affects the others), the formula

$$\frac{N_1N_2N_3}{\sqrt{(N_2^2N_3^2 + N_3^2N_1^2 + N_1^2N_2^2)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

will only be correct provided

$$\phi_1/c_1^{3/2} = \phi_2/c_2^{3/2} = \phi_3/c_3^{3/2}.$$

Even when these relations do not hold, it is shown in §§ 29, 30, 33, 34, 44 and 45, that the formula (A) is, with certain modifications and restrictions to suit particular cases (§§ 33, 34, 45), sufficiently accurate for practical purposes.

The formula (A) may be extended to any number of disturbing elements. If, for example, there were four, and the speeds corresponding to them be N_1, N_2, N_3, N_4, then the resulting whirling speed is

$$\frac{N_1N_2N_3N_4}{\sqrt{(N_1^2N_2^2N_3^2 + N_2^2N_3^2N_4^2 + N_3^2N_4^2N_1^2 + N_4^2N_1^2N_2^2)}},$$

and so on.

Considering the case of two disturbing elements, if their speeds of whirling, taken separately, be each equal to N, the resulting whirling speed due to two causes combined is $N/\sqrt{2}$.

If there were three disturbing elements, and if their speeds of whirling were all equal to N, the resulting whirling speed would be $N/\sqrt{3}$.

Of two disturbing elements, if the whirling speed for one of them be four times that of the other, the resulting whirling speed is not more than three per cent. less than the smaller whirling speed.

Concluding Remarks.

63. In conclusion, it should be noticed that in finding the speed at which a continuous shaft of given diameter, supported on bearings placed at equal distances apart, and loaded with pulleys on any or all of the spans, will whirl, the first step is to find the span which will have the biggest whirl (that is to say, the span which carries the heaviest and most advantageously situated pulleys as regards whirling), and to consider this span and the spans immediately adjacent to it on either side. The span in question can, in general, be determined on, at a glance, from the consideration of the weights, sizes, and positions of the pulleys which each span carries. Having fixed upon the three spans, the next step is to find (by the formula for each case) the whirling speed for the shaft and each of the pulleys on the three spans in question, on the assumption that the effect of every cause, except the one

under discussion, is neglected. The resulting whirling speed may then be obtained by an empirical formula of the form given in the preceding article. The speed thus obtained will be less than the actual speed of whirl (see § 46, p. 334). A nearer approximation to the actual speed might be obtained by considering only those pulleys which lie near the centres, or between the centres of the side spans and the bearings of the middle span (see § 45, p. 334), neglecting the effect of those pulleys which lie beyond the centres of the side spans. In doing so, however, there is a danger of the calculated speed *exceeding* the actual, whilst, by taking *all* the pulleys on the two sides into account, the calculated speed will be *less* than the actual speed (see § 46, p. 334).

[The above method of solution and the consideration of only three adjacent spans, is based on the results arrived at in §§ 54, 58, pp. 347, 353. It has been verified, not only by experiments made with the experimental apparatus, but also by experiments made on actual cases of shafting carrying heavy pulleys.]

In the case of a continuous shaft of equal spans which are all similarly loaded, each span whirls independently of the rest, and the problem, therefore, reduces to the case of a shaft loaded in a given manner and merely resting on a bearing at each end—the distance between the bearings being the same as between those of the continuous shaft.

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